

# Small cancellation theory and Burnside problem.

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## Abstract

In these notes we detail the geometrical approach of small cancellation theory used by T. Delzant and M. Gromov to provide a new proof of the infiniteness of free Burnside groups and periodic quotients of torsion-free hyperbolic groups.

## Contents

|          |                                                          |           |
|----------|----------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction</b>                                      | <b>2</b>  |
| 1.1      | Usual small cancellation theory. . . . .                 | 3         |
| 1.2      | Small cancellation theory in hyperbolic groups . . . . . | 3         |
| 1.3      | Iterating small cancellation . . . . .                   | 5         |
| 1.4      | Outline of the paper . . . . .                           | 6         |
| <b>2</b> | <b>Hyperbolic geometry</b>                               | <b>7</b>  |
| 2.1      | The four points inequality. . . . .                      | 7         |
| 2.2      | Quasi-geodesics . . . . .                                | 8         |
| 2.3      | Quasi-convex subsets . . . . .                           | 10        |
| 2.4      | Ultra-limit of hyperbolic spaces . . . . .               | 12        |
| 2.5      | Isometries of a hyperbolic space . . . . .               | 13        |
| 2.6      | Group acting on a hyperbolic space . . . . .             | 17        |
| <b>3</b> | <b>Rotation families</b>                                 | <b>22</b> |
| 3.1      | Fundamental theorem . . . . .                            | 22        |
| 3.2      | Consequences . . . . .                                   | 28        |
| <b>4</b> | <b>Cone over a metric space</b>                          | <b>32</b> |
| 4.1      | Definition and metric. . . . .                           | 32        |
| 4.2      | Hyperbolicity of a cone. . . . .                         | 33        |
| 4.3      | Group action on a cone . . . . .                         | 34        |
| <b>5</b> | <b>Cone-off construction</b>                             | <b>34</b> |
| 5.1      | Metric on the cone-off . . . . .                         | 35        |
| 5.2      | Uniform approximation of the distance . . . . .          | 37        |
| 5.3      | Ultra-limit and cone-off . . . . .                       | 38        |
| 5.4      | Hyperbolicity of the cone-off . . . . .                  | 44        |

|          |                                                           |           |
|----------|-----------------------------------------------------------|-----------|
| <b>6</b> | <b>Small cancellation theory</b>                          | <b>45</b> |
| 6.1      | General framework . . . . .                               | 45        |
| 6.2      | Isometries of the quotient . . . . .                      | 50        |
| 6.3      | Groups without even-torsion . . . . .                     | 51        |
| <b>7</b> | <b>Applications</b>                                       | <b>54</b> |
| 7.1      | Periodic quotients of hyperbolic groups . . . . .         | 54        |
| 7.2      | A few words about Gromov's monster group . . . . .        | 58        |
| <b>A</b> | <b>Appendix: Cartan-Hadamard Theorem</b>                  | <b>62</b> |
| A.1      | Following paths. Definition and first properties. . . . . | 63        |
| A.2      | Transitivity of the relation . . . . .                    | 65        |
| A.3      | Existence of following quasi-geodesics . . . . .          | 67        |
| A.4      | The space of quasi-geodesic paths. . . . .                | 71        |
| A.5      | Global hyperbolicity of $X$ . . . . .                     | 73        |

## 1 Introduction

Let  $n$  be an integer. A group  $G$  has exponent  $n$  if for all  $g \in G$ ,  $g^n = 1$ . In 1902 W. Burnside asked whether a finitely generated group with finite exponent is necessarily finite or not [5]. To study this question it is natural to look at the free Burnside group  $\mathbf{B}_r(n) = \mathbf{F}_r / \mathbf{F}_r^n$ , which is the quotient of the free group of rank  $r$  by the (normal) subgroup generated by the  $n$ -th powers of all elements. It is indeed the largest group of rank  $r$  and exponent  $n$ . In 1968, P. S. Novikov and S. I. Adian answered negatively the Burnside Problem. They proved that if  $r \geq 2$  and  $n$  is odd larger than 4381, then  $\mathbf{B}_r(n)$  is infinite [20]. Since, this result has been extended in many directions (even exponents, periodic quotient of hyperbolic groups, etc) [22, 24, 15, 18]. More recently T. Delzant and M. Gromov provided a new approach of the Burnside problem [10]. In particular, they give an alternative proof of the following theorem.

**Theorem 1.1** ([10, Th. 6.2.2], see also [24]). *Let  $G$  be a non-cyclic, torsion-free hyperbolic group. There exists an integer  $n_0$  such that for all odd exponents  $n \geq n_0$ , the quotient  $G/G^n$  is infinite.*

The aim of these notes is to give a comprehensive presentation of their method. Actually a more general statement holds for an arbitrary hyperbolic group.

**Theorem 1.2** ([16, Th. A]). *Let  $G$  be a non-virtually cyclic hyperbolic group. For every integer  $n_0$  there exists an exponent  $n \geq n_0$  such that the quotient  $G/G^n$  is infinite.*

However this second result require to control the even torsion, which is much harder. The notes do not cover this theorem.

From a geometrical point of view the difficulty to study Burnside groups comes from the fact that we have no “nice” metric space on which  $\mathbf{B}_r(n)$  is acting by isometries. Therefore the idea is to study  $\mathbf{B}_r(n)$  as the direct limit of a sequence of groups,

$$\mathbf{F}_r = G_0 \twoheadrightarrow G_1 \twoheadrightarrow G_3 \twoheadrightarrow G_4 \twoheadrightarrow \cdots \twoheadrightarrow G_k \twoheadrightarrow G_{k+1} \twoheadrightarrow \cdots$$

where each  $G_k$  is easier to understand. In this construction, T. Delzant and M. Gromov obtain  $G_{k+1}$  as a quotient of  $G_k$  using a geometrical version of the small cancellation theory. It forces the groups  $G_k$  to be non-virtually cyclic and hyperbolic. Thus their limit  $\mathbf{B}_r(n)$  cannot be finite. All the other known strategies also use such a sequence of groups. Before explaining the construction of T. Delzant and M. Gromov let us recall a few ideas about the usual small cancellation theory and its geometric generalization given by M. Gromov [13].

## 1.1 Usual small cancellation theory.

For more details about the usual small cancellation theory, we refer the reader to [17]. Let  $\mathbf{F}(S)$  be the free group generated by a finite set  $S$ . Let  $R$  be a set of words over the alphabet  $S \cup S^{-1}$ . The goal is to study the group  $\bar{G} = \mathbf{F}(S)/\langle\langle R \rangle\rangle$ , where  $\langle\langle R \rangle\rangle$  stands for the normal subgroup of  $\mathbf{F}(S)$  generated by  $R$ . We assume that the elements of  $R$  are non-trivial and cyclically reduced. We denote by  $R^*$  the set of all cyclic conjugates of elements of  $R \cup R^{-1}$ . A *piece* is a common prefix of two distinct elements of  $R^*$ . In other words a piece is a subword that could cancel in the product  $rs$  where  $r, s \in R^*$ . Let  $\lambda > 0$ . One says that  $R$  satisfies the small cancellation assumption  $C'(\lambda)$  if for all pieces  $u$ , for all relations  $r \in R$  containing  $u$ ,  $|u| \leq \lambda|r|$ . Let us mention one important theorem. In the next paragraph we will provide an extension of it.

**Theorem 1.3** (see [11]). *Let  $\lambda \in (0, 1/6)$ . Let  $R$  be a set of non-trivial, cyclically reduced words over the alphabet  $S \cup S^{-1}$ . If  $R$  satisfies the condition  $C'(\lambda)$  then the quotient  $\mathbf{F}(S)/\langle\langle R \rangle\rangle$  is a hyperbolic group.*

For our purpose we are going to consider a more stronger condition called  $C''(\lambda)$ . It requires that for all pieces  $u$ , for all relations  $r \in R$  (not necessarily containing  $u$ ),  $|u| \leq \lambda|r|$ . This assumption can be reformulated in a more geometrical way. To that end we consider the Cayley graph of  $\mathbf{F}(S)$  with respect to  $S$ , denoted by  $X$  which is a tree. Let  $r \in R^*$ . It fixes two points  $r^-$  and  $r^+$  of the boundary at infinity  $\partial X$  of  $X$ . These points are joined by a bi-infinite geodesic called the *axis* of  $r$ . We denote it by  $Y_r$ . The isometry  $r$  acts on  $Y_r$  by translation of length  $[r] = \inf_{x \in X} |rx - x|$ , where  $|x - y|$  denotes the distance between  $x$  and  $y$ . Since  $r$  is a cyclically reduced word,  $[r]$  is in fact the length of the relation  $r$ . Consider now two relations  $r, s \in R^*$ . The length of a common piece of  $r$  and  $s$  is exactly  $\text{diam}(Y_r \cap Y_s)$ . Thus the  $C''(\lambda)$  condition can be stated in this way:

$$\sup_{r \neq s} \text{diam}(Y_r \cap Y_s) \leq \lambda \inf_r [r].$$

With this idea in mind we provide in the next paragraph a larger framework to the small cancellation theory.

## 1.2 Small cancellation theory in hyperbolic groups

From now on  $X$  is a proper geodesic  $\delta$ -hyperbolic space. Let  $G$  be a group acting properly co-compactly by isometries on  $X$ . The translation length of an isometry  $g \in G$ , denoted by  $[g]$  is the quantity  $[g] = \inf_{x \in X} |gx - x|$ . We consider a collection  $\mathcal{Q}$  of pairs  $(H, Y)$  satisfying the following properties

- (i)  $Y$  is a  $2\delta$ -quasi-convex subset of  $X$  and  $H$  a subgroup of  $\text{Stab}(Y)$ , the stabilizer in  $G$  of  $Y$ , acting co-compactly on  $Y$ .

- (ii) the family  $\mathcal{Q}$  is invariant under the action of  $G$  i.e., for all  $g \in G$ , for all  $(H, Y) \in \mathcal{Q}$ ,  $(gHg^{-1}, gY)$  is an element of  $\mathcal{Q}$ ,
- (iii) the set  $\mathcal{Q}/G$  is finite.

Our goal is to study the group  $\bar{G} = G/K$ , where  $K$  is the (normal) subgroup of  $G$  generated by the  $H$ 's. For this purpose, we define two quantities which respectively play the role of the length of the largest piece and the length of the smallest relation.

- The maximal overlap between two quasi-convex of  $\mathcal{Q}$  is measured by the quantity

$$\Delta(\mathcal{Q}) = \sup_{(H_1, Y_1) \neq (H_2, Y_2)} \text{diam}(Y_1^{+5\delta} \cap Y_2^{+5\delta}).$$

- The smallest translation length of  $\mathcal{Q}$  is defined as

$$T(\mathcal{Q}) = \inf \{ |h| \mid h \in H \setminus \{1\}, (H, Y) \in \mathcal{Q} \}.$$

Note that if  $\Delta(\mathcal{Q})$  is finite then two distinct groups  $H$  cannot be associated in  $\mathcal{Q}$  to the same quasi-convex  $Y$ . In particular  $H$  is a normal subgroup of  $\text{Stab}(Y)$ . We are interested in situations where the ratios  $\Delta(\mathcal{Q})/T(\mathcal{Q})$  and  $\delta/T(\mathcal{Q})$  are very small (see Theorem 1.4). In particular it forces every non-trivial element of every subgroup  $H$  to be hyperbolic.

### Examples.

- (i) Let  $R$  be a finite set of elements of  $\mathbf{F}(S)$ . Consider the collection

$$\mathcal{Q} = \{ (\langle uru^{-1} \rangle, uY_r) \mid r \in R, u \in \mathbf{F}(S) \}.$$

In this case we recover the usual small cancellation theory. Since the the Cayley graph of  $\mathbf{F}(S)$  is 0-hyperbolic, the assumption  $C''(\lambda)$  is equivalent to  $\Delta(\mathcal{Q}) \leq \lambda T(\mathcal{Q})$

- (ii) The next example comes from small cancellation with graphs (see [21]). Let  $\Gamma$  be a finite connected graph whose edges are labelled with elements of  $S \cup S^{-1}$ . Given a vertex  $v_0$  of  $\Gamma$ , the labeling provides a homomorphism from the fundamental group of  $(\Gamma, v_0)$  onto a subgroup  $H$  of  $\mathbf{F}(S)$ . Moreover, we have a map from  $\tilde{\Gamma}$ , the universal cover of  $\Gamma$ , onto a subtree  $T$  of the Cayley graph of  $\mathbf{F}(S)$ . We can consider the collection

$$\mathcal{Q} = \{ (gHg^{-1}, gT) \mid g \in G/H \}.$$

In this situation a *piece* is, by definition, a word which labels two distinct simple paths of  $\Gamma$ . Then  $\Delta(\mathcal{Q})$  is the length of the largest piece whereas  $T(\mathcal{Q})$  is the girth of the graph  $\Gamma$  i.e., the length of the smallest embedded loop in  $\Gamma$ . For further detail we refer the reader to [14, 1/6-theorem] and [21, Th.1].

- (iii) Let  $G$  be a group acting properly, co-compactly by isometries on a geodesic,  $\delta$ -hyperbolic space  $X$ . If  $r \in G$  is a hyperbolic isometry it fixes two points  $r^-$  and  $r^+$  of  $\partial X$ , the boundary at infinity of  $X$ . Following the case of free groups we define  $Y_r$  to be the set of all points of  $X$  which are  $10\delta$ -close to some bi-infinite geodesic joining  $r^-$  and  $r^+$ . This set is  $2\delta$ -quasi-convex.

We consider the set  $P$  of all hyperbolic elements  $r \in G$  which are not proper powers such that  $[r] \leq 1000\delta$ . To every integer  $n$  we associate the collection

$$\mathcal{Q}_n = \{(\langle r^n \rangle, Y_r) \mid r \in P\}$$

Assume now that  $G$  is non-virtually cyclic. Since  $G$  acts properly co-compactly on  $X$ , the overlap  $\Delta(\mathcal{Q}_n)$  is uniformly bounded. Therefore, by choosing an integer  $n$  large enough, one can obtain a ratio  $\Delta(\mathcal{Q}_n)/T(\mathcal{Q}_n)$  as small as desired.

Consider now an arbitrary family  $\mathcal{Q}$  as defined previously. In order to study the quotient  $\bar{G} = G/K$  we construct a space  $\bar{X}$  on which  $\bar{G}$  acts properly, co-compactly, by isometries. Roughly speaking,  $\bar{X}$  is obtained as follows. We fix a large real number  $\rho$ . Its value will be made precise later. For each  $(H, Y) \in \mathcal{Q}$ , we start by attaching on  $X$  along  $Y$  a cone of base  $Y$  and radius  $\rho$ . The cones are endowed with a negatively curved metric modeled on the distance of the hyperbolic plane  $\mathbf{H}_2$ . This new space  $\check{X}$  (called cone-off of  $X$ ) inherits a metric coming from the distance on  $X$  and the ones on the cones. Since  $\mathcal{Q}$  is endowed with an action of  $G$ , the action of  $G$  on  $X$  extends by homogeneity into an action of  $G$  on  $\check{X}$ . On this space every group  $H$  acts as a rotation of very large angle that fixes the apex of the cone of base  $Y$ . One speaks of *rotation family* (see Section 3). The space  $\bar{X}$  is then the quotient of  $\check{X}$  by  $K$ . By construction  $\bar{X}$  is proper and geodesic. Moreover,  $\bar{G}$  acts properly co-compactly by isometries on it. The study of the metric on  $\check{X}$  and  $\bar{X}$  is a key question. This will be done in Sections 4-6. One of the main features of the space  $\bar{X}$  is the following. Assume that the ratio  $\Delta(\mathcal{Q})/T(\mathcal{Q})$  is very small, then every ball of  $\bar{X}$  of radius  $\rho/50$  is roughly  $\delta$ -hyperbolic where  $\delta$  is the hyperbolicity constant of the hyperbolic plane  $\mathbf{H}_2$ . If  $\rho$  has been chosen large enough, we can then apply a Cartan-Hadamard Theorem (see Theorem A.1). It leads to the following analogue of Theorem 1.3.

**Theorem 1.4.** *There exist positive constants  $\delta_0$  and  $\lambda_0$  (which do not depend on  $X$ ,  $G$  or  $\mathcal{Q}$ ) with the following property. Assume that*

$$\frac{\delta}{T(\mathcal{Q})} \leq \delta_0 \quad \text{and} \quad \frac{\Delta(\mathcal{Q})}{T(\mathcal{Q})} \leq \lambda_0.$$

*Then  $\bar{X}$  is a hyperbolic space endowed with a proper co-compact action of  $\bar{G}$ .*

Some hypotheses of the theorem can be weaken. For instance we do not really need that the action of  $G$  (respectively  $H$ ) on  $X$  (respectively  $Y$ ) to be co-compact. Without these assumptions, the space  $\bar{X}$  remains hyperbolic nevertheless the action of  $\bar{G}$  on  $\bar{X}$  is no more proper or co-compact.

### 1.3 Iterating small cancellation

Theorem 1.4 is an important step of the induction process to prove the infiniteness of the Burnside groups. However this is definitely not sufficient. The space  $\bar{X}$  has in fact many other properties that we will enlighten in Section 6. Let us explain briefly where the difficulty does come from. Let  $G_0$  be a non-virtually cyclic torsion-free group acting properly co-compactly on a  $\delta$ -hyperbolic space  $X_0$  and  $n_0$  an integer. Following Example (iii), we consider the quotient  $G_1 = G_0/K_0$  where  $K_0$  is the normal subgroup of  $G_0$  generated by

$$\{r^{n_0}/r \in G, \text{ hyperbolic, not a proper power, } [r] \leq 1000\delta\}$$

If  $n_0$  is large enough, Theorem 1.4 applies. The quotient  $G_1$  is also a hyperbolic group. Thus we would like to iterate the process and kill some large powers in  $G_1$ . According to Example (iii) this can be done, but there is no reason that the exponent  $n_1$  that works for the second step should be the same as  $n_0$ . In this way one can construct by induction an infinite torsion group. Nevertheless to study Burnside groups we need to follow during the construction some parameters that allow us to use at each step the same exponent  $n$ . Note already that the constants  $\delta_0$  and  $\lambda_0$  in Theorem 1.4 do not depend on  $G$ ,  $X$  or  $\mathcal{Q}$ . This is an important but not sufficient fact. To control the ratio  $\Delta(\mathcal{Q})/T(\mathcal{Q})$  we will take care of two quantities associated to the action of  $G$  on  $X$ . The radius of injectivity which represent the smallest asymptotic translation length of a hyperbolic element of  $G$  and the invariant  $A(G, X)$  which measures the maximal overlap between the axes of two small isometries of  $G$  (see Section 2.6 for the precise definitions). During this construction we will require the exponent  $n$  to be odd. With this assumption we can prove that every subgroup of any  $G_k$  is either cyclic or contains a free group. It will help to control  $\Delta(\mathcal{Q})$  in terms of  $A(G, X)$ . Burnside groups of large even exponents are known to be infinite [15, 18]. However the proof is much harder.

The hypotheses required to apply small cancellation are not very restricting. See for instance [25]. Therefore iterating small cancellation can be exploited to build many groups with pathological properties. It is one of the ingredients involved in the construction of Gromov's monster group [14, 1]. This object is a limit of groups  $G_k$  where  $G_{k+1}$  is obtained from  $G_k$  using graphical small cancellation (see Example (ii)). One remarkable property of this group is that it does not coarsely embed into a Hilbert space and thus does not satisfy the Baum-Connes conjecture with coefficients.

For other approaches of iterated small cancellation theory we refer the reader to [27, 23, 19]

## 1.4 Outline of the paper

In the Section 2 we recall the main features of hyperbolic spaces in the sense of Gromov and groups acting on a hyperbolic space. Section 3 is dedicated to the study of rotation families. This section differs from the tools used by T. Delzant and M. Gromov. In this way we do not need to use orbifolds as in [10]. Sections 4 and 5 detail the cone-off construction. The core of the proof is contained in Section 6. We prove among others the small cancellation theorem (see Theorem 6.11) and investigate the properties of the space  $\bar{X}$ . Thanks to the small cancellation we state at the beginning of Section 7.1 an induction lemma which explains how the group  $G_{k+1}$  can be obtained from  $G_k$ . In particular it provides a set of assumptions which if they are satisfied for  $G_k$  also hold for  $G_{k+1}$ . This allow us to iterate the construction and prove the main theorem (see Theorem 7.7). Our approach of the small cancellation theory is very broad. In particular it covers all the main results that are needed to construct Gromov's monster group. In Section 7.2 we also give some keys to understand this construction as explained in [1]. In the appendix, we propose an alternative proof of the Cartan-Hadamard Theorem based on the ideas given in [10].

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## 2 Hyperbolic geometry

In this section we some of the standard facts on hyperbolic spaces in the sense of M. Gromov. We only give the proofs of quantitative results, which are not a straight forward application of the four points condition. For more details we refer the reader to the original paper of M. Gromov [12] or to [6, 11].

Let  $X$  be a metric length space. Unless otherwise stated a path is a rectifiable path parametrized by arclength. Given two points  $x$  and  $x'$  of  $X$ , we denote by  $|x - x'|_X$  (or simply  $|x - x'|$ ) the distance between them. We write  $B(x, r)$  for the closed ball of center  $x$  and radius  $r$  i.e., the set of points  $y \in X$  such that  $|x - y| \leq r$ . Let  $Y$  be a subset of  $X$ . The distance between a point  $x$  of  $X$  and  $Y$  is denoted by  $d(x, Y)$ . We write  $Y^{+\alpha}$  for the  $\alpha$ -neighborhood of  $Y$  i.e., the set of points  $x \in X$  such that  $d(x, Y) \leq \alpha$ . Let  $\eta \geq 0$ . A point  $p$  of  $Y$  is an  $\eta$ -projection of  $x \in X$  on  $Y$  if  $|x - p| \leq d(x, Y) + \eta$ . A 0-projection is simply called a *projection*.

### 2.1 The four points inequality.

The *Gromov product* of three points  $x, y, z \in X$  is defined by

$$\langle x, y \rangle_z = \frac{1}{2} \left\{ |x - z| + |y - z| - |x - y| \right\}.$$

The space  $X$  is  $\delta$ -hyperbolic if for every  $x, y, z, t \in X$

$$\langle x, z \rangle_t \geq \min \left\{ \langle x, y \rangle_t, \langle y, z \rangle_t \right\} - \delta, \quad (1)$$

or equivalently

$$|x - z| + |y - t| \leq \max \left\{ |x - y| + |z - t|, |y - z| + |x - t| \right\} + 2\delta. \quad (2)$$

To show that a space is hyperbolic, it is actually sufficient to prove (1) with a fixed base point.

**Lemma 2.1** (see [6, Chap.1 Prop. 1.2]). *Let  $\delta \geq 0$ . Let  $(X, t)$  be a pointed metric space. If for every  $x, y, z \in X$ , we have  $\langle x, z \rangle_t \geq \min \{ \langle x, y \rangle_t, \langle y, z \rangle_t \} - \delta$ , then  $X$  is  $2\delta$ -hyperbolic.*

**Remarks.** Note that in the definition of hyperbolicity we do not assume that  $X$  is a geodesic. In Section 5 we construct a length space (the cone-off) for which it is not obvious to prove that it is geodesic. However it satisfies (1). Therefore we prefer to state all the results concerning hyperbolicity for the class of length spaces. To compensate for the absence of geodesics we use the following property. For every  $x, x' \in X$ , for every  $\eta > 0$ , for every  $0 \leq l \leq |x - x'|$  there exists  $y \in X$  such that  $|x - y| = l$  and  $\langle x, x' \rangle_y \leq \eta$ . In this context, the Gromov product  $\langle x, x' \rangle_y$  should be thought as an analogue of the distance between  $y$  and a “geodesic” joining  $x$  and  $x'$ .

If  $X$  is 0-hyperbolic, then it can be isometrically embedded in an  $\mathbf{R}$ -tree [11, Chap. 2, Prop. 6]. However we will always assume that the hyperbolicity constant  $\delta$  is positive. Most of the results hold for  $\delta = 0$ . But this is more convenient to define particular subsets (see for instance Definition 2.14 or Lemmas 2.12 and 2.13) without introducing other auxiliary positive parameters.

The hyperbolicity constant of the hyperbolic plane  $\mathbf{H}_2$  will play a particular role. Therefore we denote it by  $\delta$  (bold delta).

From now on we assume that the space  $X$  is  $\delta$ -hyperbolic. It is known that triangles in a hyperbolic geodesic space are thin (every side lies in a uniform neighborhood of the union of the two others). Since our space is not necessarily geodesic, we use instead the following metric inequalities. They will be used in situations where the Gromov products  $\langle x, y \rangle_s$ ,  $\langle x, y \rangle_t$  or  $\langle x, z \rangle_t$  are “small”. Their proof is left to the reader.

**Lemma 2.2.** *Let  $x, y, z, s$  and  $t$  be five points of  $X$ .*

- (i)  $\langle x, y \rangle_t \leq \max \left\{ |x - t| - \langle y, z \rangle_x, \langle x, z \rangle_t \right\} + \delta$ ,
- (ii)  $|s - t| \leq \left| |x - s| - |x - t| \right| + 2 \max \left\{ \langle x, y \rangle_s, \langle x, y \rangle_t \right\} + 2\delta$ ,
- (iii) *The distance  $|s - t|$  is bounded above by*

$$\max \left\{ \left| |x - s| - |x - t| \right| + 2 \max \left\{ \langle x, y \rangle_s, \langle x, z \rangle_t \right\}, |x - s| + |x - t| - 2 \langle y, z \rangle_x \right\} + 4\delta.$$

## 2.2 Quasi-geodesics

**Definition 2.3.** Let  $l \geq 0$ ,  $k \geq 1$  and  $L \geq 0$ . Let  $I$  be an interval of  $\mathbf{R}$ . A path  $\gamma : I \rightarrow X$  is

- (i) a  $(k, l)$ -quasi-geodesic if for all  $t, t' \in I$

$$|\gamma(t) - \gamma(t')| \leq |t - t'| \leq k |\gamma(t) - \gamma(t')| + l.$$

- (ii) an  $L$ -local  $(k, l)$ -quasi-geodesic if the restriction of  $\gamma$  to any interval of length  $L$  is a  $(k, l)$ -quasi-geodesic.

**Remarks.** The first inequality in the definition just follows from the fact that  $\gamma$  is parametrized by arc length. Since  $X$  is a length space, for every  $x, x' \in X$ , for every  $l > 0$ , there exists a  $(1, l)$ -quasi-geodesic joining  $x$  and  $x'$  (take a rectifiable path between them whose length is shorter than  $|x - x'| + l$ ). Let  $\gamma : I \rightarrow X$  be a  $(1, l)$ -quasi-geodesic. Let  $x = \gamma(t)$ ,  $x' = \gamma(t')$  and  $y = \gamma(s)$  be three points on  $\gamma$ . If  $t \leq s \leq t'$  then  $\langle x, x' \rangle_y \leq l/2$ .

**Proposition 2.4.** *Let  $\gamma : I \rightarrow X$  be a  $(1, l)$ -quasi-geodesic of  $X$ .*

- (i) *Let  $x$  be a point of  $X$  and  $p$  an  $\eta$ -projection of  $x$  on  $\gamma$ . For all  $y \in \gamma$ ,  $\langle x, y \rangle_p \leq l + \eta + 2\delta$ .*
- (ii) *For every  $x \in X$  for every  $y, y' \in \gamma$  we have  $\langle y, y' \rangle_x - l \leq d(x, \gamma) \leq \langle y, y' \rangle_x + l + 3\delta$ .*

*Proof.* Let  $x$  be a point of  $X$ . We denote by  $p = \gamma(s)$  an  $\eta$ -projection of  $x$  on  $\gamma$  and  $y = \gamma(t)$  a point of  $\gamma$ . By reversing if necessary the parametrization of  $\gamma$  we can assume that  $s \leq t$ . Note that  $|y - p| \geq \langle x, y \rangle_p$ . Therefore there exists  $r \in [s, t]$  such that the point  $z = \gamma(r)$  satisfies  $|z - p| = \langle x, y \rangle_p$  and  $\langle p, y \rangle_z \leq l/2$ . By Lemma 2.2-(i) we have  $\langle x, p \rangle_z \leq l/2 + \delta$  i.e.,

$$\langle x, y \rangle_p = |p - z| \leq |x - p| - |x - z| + l + 2\delta.$$

Nevertheless,  $p$  is an  $\eta$ -projection of  $x$  on  $\gamma$ . Thus  $|x - p| - |x - z| \leq \eta$ . It follows that  $\langle x, y \rangle_p \leq l + \eta + 2\delta$ , which proves (i).



Let  $x \in X$ . Let  $y$  and  $y'$  be two points of  $\gamma$ . The first inequality of (ii) follows from the triangle inequality. We denote by  $p$  a projection of  $x$  on  $\gamma$ . It follows from Point (i) that  $d(x, \gamma) = \langle y, p \rangle_x + \langle x, y \rangle_p$  is bounded above by  $\langle y, p \rangle_x + l + 2\delta$ . In the same way  $d(x, \gamma) \leq \langle y', p \rangle_x + l + 2\delta$ . Consequently, the hyperbolicity condition (1) gives

$$d(x, \gamma) \leq \min \{ \langle y, p \rangle_x, \langle y', p \rangle_x \} + l + 2\delta \leq \langle y, y' \rangle_x + l + 3\delta. \quad \square$$

**Proposition 2.5** (Stability of quasi-geodesics [6, Chap. 3, Th. 1.2-1.4]). *Let  $k \geq 1$ ,  $k' > k$  and  $l \geq 0$ . There exists  $L \geq 0$ ,  $D \geq 0$  which only depend on  $\delta$ ,  $k$ ,  $k'$  and  $l$  such that the Hausdorff distance between two  $L$ -local  $(k, l)$ -quasi-geodesics joining the same endpoints is at most  $D$ . Moreover, every  $L$ -local  $(k, l)$ -quasi-geodesic is also a (global)  $(k', l)$ -quasi-geodesic.*

In this paper we are mostly using  $L$ -local  $(1, l)$ -quasi-geodesics. For these paths one can easily provide a precise value for  $D$  (see Corollary 2.6). This is not really crucial but it will decrease the number of parameters that we have to deal with in all the proofs.

**Corollary 2.6.** *Let  $l, l' \geq 0$ . There exists  $L = L(l, l', \delta)$  which only depends on  $\delta$ ,  $l$  and  $l'$  with the following property. Let  $\gamma : [a, b] \rightarrow X$  be a  $L$ -local  $(1, l)$ -quasi-geodesic and  $\gamma' : [a', b'] \rightarrow X$  a  $L$ -local  $(1, l')$ -quasi-geodesic. If they join the same extremities then  $\gamma$  lies in the  $(l/2 + l' + 5\delta)$ -neighborhood of  $\gamma'$ .*

*Proof.* Let  $l \geq 0$ . The constants  $L$ ,  $D$  and  $k$  are chosen so that we can apply Proposition 2.5 to  $\gamma$  and  $\gamma'$ : the Hausdorff distance between  $\gamma$  and  $\gamma'$  is at most  $D$  and  $\gamma'$  is a  $(k, l')$ -global quasi-geodesic. Note that  $k$ ,  $D$  and  $L$  only depend on  $l$ ,  $l'$  and  $\delta$ . Without loss of generality we may assume that  $L > k(4D + 3l + 6\delta) + l'$ . Let  $x = \gamma(t)$  be a point on  $\gamma$ . We assume that  $|a - t|$  and  $|b - t|$  are bounded below by  $D + 3l/2 + 3\delta$ . The other cases are similar. Let us put  $t_{\pm} = t \pm (D + 3l/2 + 3\delta)$  and  $x_{\pm} = \gamma(t_{\pm})$ . By the stability of quasi-geodesics there exist points  $y_{\pm} = \gamma'(s_{\pm})$  such that  $|x_{\pm} - y_{\pm}| \leq D$ . Without loss of generality we can assume that  $s_- \leq s_+$ . By hyperbolicity we obtain

$$\min \{ |x_- - x| - |x_- - y_-|, \langle y_-, y_+ \rangle_x, |x_+ - x| - |x_+ - y_+| \} \leq \langle x_-, x_+ \rangle_x + 2\delta \leq l/2 + 2\delta.$$

However by construction  $|x_{\pm} - x|$  is bounded below by  $D + l/2 + 3\delta$  and  $|x_{\pm} - y_{\pm}|$  above by  $D$ . Therefore, we necessarily have  $\langle y_-, y_+ \rangle_x \leq l/2 + 2\delta$ . We now claim that  $|s_+ - s_-| \leq L$ . Indeed  $\gamma'$  is a  $(k, l')$ -quasi-geodesic. Therefore the triangle inequality leads to

$$k^{-1} (|s_+ - s_-| - l') \leq |y_+ - y_-| \leq |x_+ - x_-| + 2D \leq 4D + 3l + 6\delta$$

Our claim follows from the assumption on  $L$ . In particular  $\gamma'$  restricted to  $[s_-, s_+]$  is a (global)  $(1, l)$ -quasi-geodesic. By Proposition 2.4(ii), the point  $x$  is  $(l/2 + l' + 5\delta)$ -close to  $\gamma'$ .  $\square$

**Remarks.** We keep the notations of the corollary. Let  $x = \gamma(t)$ ,  $x' = \gamma(t')$  and  $y = \gamma(s)$  be three points on  $\gamma$ . If  $t \leq s \leq t'$  then  $\langle x, x' \rangle_y \leq l/2 + 5\delta$ . Moreover, for every  $p \in X$  we have  $d(p, \gamma) \leq \langle x, x' \rangle_p + l + 8\delta$ .

**Proposition 2.7** (Stability of discrete quasi-geodesics). *Let  $l > 0$ . There exists  $L = L(l, \delta)$  which only depends on  $\delta$  and  $l$  with the following property. Let  $x_0, \dots, x_m$  be a sequence of points of  $X$  such that*

- (i) for every  $i \in \{1, \dots, m-1\}$ ,  $\langle x_{i-1}, x_{i+1} \rangle_{x_i} \leq l$ ,
- (ii) for every  $i \in \{1, \dots, m-2\}$ ,  $|x_{i+1} - x_i| \geq L$ .

Then for all  $i \in \{0, \dots, m\}$ ,  $\langle x, x' \rangle_{x_i} \leq l + 5\delta$ . Moreover, for all  $p \in X$  there exists  $i \in \{0, \dots, m-1\}$  such that  $\langle x_{i+1}, x_i \rangle_p \leq \langle x_0, x_m \rangle_p + 2l + 8\delta$ .

*Proof.* Let  $\eta > 0$ . For every  $i \in \{0, \dots, m-1\}$  we choose a  $(1, \eta)$ -quasi-geodesic  $\gamma_i$  joining  $x_i$  to  $x_{i+1}$ . We denote by  $\gamma$  the concatenation  $\gamma_0 \dots \gamma_{m-1}$ . According to our assumptions this is a  $L$ -local  $(1, 2l + 4\eta)$ -quasi-geodesic. We can therefore apply Proposition 2.5. If  $L$  is sufficiently large then we obtain the followings.

- (i) For every  $i \in \{0, \dots, m\}$  we have  $\langle x, x' \rangle_{x_i} \leq l + 2\eta + 5\delta$ .
- (ii) For every  $p \in X$  we have  $d(p, \gamma) \leq \langle x_0, x_m \rangle_p + 2l + 4\eta + 8\delta$ . However there exists  $i \in \{0, \dots, m-1\}$  such that the distance between  $p$  and  $\gamma$  is exactly  $d(p, \gamma_i)$ . The path  $\gamma_i$  being a  $(1, \eta)$ -quasi-geodesic, we get  $\langle x_i, x_{i+1} \rangle_p \leq \langle x_0, x_m \rangle_p + 2l + 5\eta + 8\delta$ .

The inequalities that we obtained hold for every sufficiently small  $\eta > 0$ , which gives the desired conclusion.  $\square$

## 2.3 Quasi-convex subsets

**Definition 2.8.** Let  $\alpha \geq 0$ . A subset  $Y$  of  $X$  is  $\alpha$ -quasi-convex if for every  $x \in X$ , for every  $y, y' \in Y$ ,  $d(x, Y) \leq \langle y, y' \rangle_x + \alpha$ .

**Remark.** Since  $X$  is not a geodesic space our definition of quasi-convex slightly differs from the usual one (every geodesic joining two points of  $Y$  remains in the  $\alpha$ -neighborhood of  $Y$ ). However if  $X$  is geodesic, an  $\alpha$ -quasi-convex subset in the usual sense is  $(\alpha + 3\delta)$ -quasi-convex in our sense and conversely. According to Proposition 2.4 every  $(1, l)$ -quasi-geodesic is  $(l + 3\delta)$ -quasi-convex. If  $L$  is sufficiently large then every  $L$ -local  $(1, l)$ -quasi-geodesic is  $(l + 8\delta)$ -quasi-convex (see Corollary 2.6). For our purpose we will also need a slightly stronger version of quasi-convexity.

**Definition 2.9.** Let  $Y$  be a subset of  $X$  connected by rectifiable paths. The length metric on  $Y$  induced by the restriction of  $|\cdot|_X$  to  $Y$  is denoted by  $|\cdot|_Y$ . We say that  $Y$  is *strongly quasi-convex* if  $Y$  is  $2\delta$ -quasi-convex and for every  $y, y' \in Y$ ,

$$|y - y'|_X \leq |y - y'|_Y \leq |y - y'|_X + 8\delta.$$

**Remark.** The first inequality is just a consequence of the definition of  $|\cdot|_Y$ .

**Lemma 2.10** (Projection on a quasi-convex. Compare [6, Chap. 10, Prop. 2.1]). *Let  $Y$  be an  $\alpha$ -quasi-convex subset of  $X$ . Let  $x, x' \in X$ .*

- (i) *If  $p$  is an  $\eta$ -projection of  $x$  on  $Y$ , then for all  $y \in Y$ ,  $\langle x, y \rangle_p \leq \alpha + \eta$ .*
- (ii) *If  $p$  and  $p'$  are respective  $\eta$ - and  $\eta'$ -projections of  $x$  and  $x'$  on  $Y$ , then*

$$|p - p'| \leq \max \left\{ |x - x'| - |x - p| - |x' - p'| + 2\varepsilon, \varepsilon \right\},$$

where  $\varepsilon = 2\alpha + \eta + \eta' + \delta$ .

**Lemma 2.11** (Neighborhood of a quasi-convex. Compare [6, Chap. 10, Prop. 1.2]). *Let  $\alpha \geq 0$ . Let  $Y$  be an  $\alpha$ -quasi-convex subset of  $X$ . For every  $A \geq \alpha$ , the  $A$ -neighborhood of  $Y$  is  $2\delta$ -quasi-convex.*

**Lemma 2.12.** *Let  $Y_1$  and  $Y_2$  be respectively  $\alpha_1$ - and  $\alpha_2$ -quasi-convex subsets of  $X$ . The subset*

$$Z = Y_1^{+\alpha_1+3\delta} \cap Y_2^{+\alpha_2+3\delta}$$

*is  $7\delta$ -quasi-convex.*

*Proof.* Let  $x \in X$  and  $t, t' \in Z$ . Let  $\eta \in (0, 2\delta)$ . We denote by  $\gamma$  a  $(1, \eta)$ -quasi-geodesic joining  $t$  and  $t'$ . Let  $u$  be a point of  $\gamma$ . In particular  $\langle t, t' \rangle_u \leq \eta/2$ . We assume first that  $|t - u| \geq 4\delta + 2\eta$  and  $|t' - u| \geq 4\delta + 2\eta$ . We denote by  $y$  and  $y'$  respective  $\eta$ -projections of  $t$  and  $t'$  on  $Y_i$ . By hyperbolicity

$$\min \left\{ \langle t, y \rangle_u, \langle y, y' \rangle_u, \langle y', t' \rangle_u \right\} \leq \langle t, t' \rangle_u + 2\delta \leq \frac{1}{2}\eta + 2\delta. \quad (3)$$

Assume that the minimum is achieved by  $\langle t, y \rangle_u$ , it gives

$$|y - u| = |t - y| - |t - u| + 2\langle t, y \rangle_u \leq \alpha_i + 3\delta$$

Thus  $u$  belongs to the  $(\alpha_i + 3\delta)$ -neighborhood of  $Y_i$ . The same holds if the minimum in (3) is achieved by  $\langle y', t' \rangle_u$ . Suppose now that the minimum is achieved by  $\langle y, y' \rangle_u$ . The set  $Y_i$  being  $\alpha$ -quasi-convex we get  $d(u, Y_i) \leq \langle y, y' \rangle_u + \alpha_i \leq \alpha_i + 3\delta$ . In all cases  $u$  lies in the  $(\alpha_i + 3\delta)$ -neighborhood of  $Y_i$ . It follows that any point of  $\gamma$  is in the  $(4\delta + 2\eta)$ -neighborhood of  $Z$ .

Let  $p$  be a projection of  $x$  on  $\gamma$ . The path  $\gamma$  is  $(\eta + 3\delta)$ -quasi-convex (Proposition 2.4) hence  $|x - p| \leq \langle t, t' \rangle_x + \eta + 3\delta$ . According to the previous remark  $p$  lies in the  $(4\delta + 2\eta)$ -neighborhood of  $Z$  thus  $d(x, Z) \leq \langle t, t' \rangle_x + 3\eta + 7\delta$ . This inequality holds for every sufficiently small  $\eta$ . Thus  $Z$  is  $7\delta$ -quasi-convex.  $\square$

**Lemma 2.13.** *Let  $Y$  and  $Z$  be respectively  $\alpha$ - and  $\beta$ -quasi-convex subsets of  $X$ . For all  $A \geq 0$  we have*

$$\text{diam}(Y^{+A} \cap Z^{+A}) \leq \text{diam}(Y^{+\alpha+3\delta} \cap Z^{+\beta+3\delta}) + 2A + 4\delta.$$

*Proof.* Let  $x$  and  $x'$  be two points of  $Y^{+A} \cap Z^{+A}$ . We assume that  $|x - x'| > 2A + 4\delta$ . Let  $\eta \in (0, \delta)$  such that  $|x - x'| > 2A + 4\delta + 6\eta$ . There exist  $t, t' \in X$  such that  $|x - t| = |x' - t'| = A + 2\delta + 3\eta$  and  $\langle x, x' \rangle_t, \langle x, x' \rangle_{t'} \leq \eta$ . Note that  $|x - t'|, |x' - t| \geq A + 2\delta + 3\eta$ . We claim that  $t$  and  $t'$  belong to the  $(\alpha + 3\delta)$ -neighborhood of  $Y$ . Let us denote by  $y$  and  $y'$  respective  $\eta$ -projection of  $x$  and  $x'$  on  $Y$ . By hyperbolicity

$$\min \{ |x - t| - |x - y|, \langle y, y' \rangle_t, |x' - t| - |x' - y'| \} \leq \langle x, x' \rangle_t + 2\delta \leq 2\delta + \eta.$$

It follows that  $\langle y, y' \rangle_t \leq 2\delta + \eta \leq 3\delta$ . The subset  $Y$  being  $\alpha$ -quasi-convex we get  $d(t, Y) \leq \alpha + 3\delta$ . The same holds for  $t'$ , which proves our claim. Similarly  $t$  and  $t'$  lie in the  $(\beta + 3\delta)$ -neighborhood of  $Z$ . Consequently,

$$|x - x'| \leq |t - t'| + 2A + 4\delta + 6\eta \leq \text{diam}(Y^{+\alpha+3\delta} \cap Z^{+\beta+3\delta}) + 2A + 4\delta + 6\eta.$$

This last inequality actually holds for every sufficiently small  $\eta$  and  $x, x'$  in  $Y^{+A} \cap Z^{+A}$ , which leads to the conclusion.  $\square$

**Definition 2.14.** Let  $Y$  be a subset of  $X$ . The *hull* of  $Y$  denoted by  $\text{hull}(Y)$  is the union of all  $(1, \delta)$ -quasi-geodesics joining two points of  $Y$ .

**Lemma 2.15.** Let  $Y$  be a subset of  $X$ . The hull of  $Y$  is  $6\delta$ -quasi-convex.

*Proof.* Let  $x \in X$  and  $y, y' \in \text{hull}(Y)$ . By definition there exist  $\gamma : [a, b] \rightarrow X$  and  $\gamma' : [a', b'] \rightarrow X$  two  $(1, \delta)$ -quasi-geodesics joining points of  $Y$  such that  $y$  and  $y'$  respectively lie on  $\gamma$  and  $\gamma'$ . Since  $X$  is a length space, there exists a  $(1, \delta)$ -quasi-geodesic  $\gamma_0$  between  $\gamma(a)$  and  $\gamma'(a')$ . In particular  $\gamma_0 \subset \text{hull}(Y)$ . By hyperbolicity

$$\min \left\{ \langle y, \gamma(a) \rangle_x, \langle \gamma(a), \gamma(a') \rangle_x, \langle \gamma(a'), y' \rangle_x \right\} \leq \langle y, y' \rangle_x + 2\delta.$$

However  $\gamma$  is  $4\delta$ -quasi-convex (Proposition 2.4), thus  $d(x, \text{hull}(Y)) \leq d(x, \gamma) \leq \langle y, \gamma(a) \rangle_x + 4\delta$ . We have similar inequalities for  $\gamma_0$  and  $\gamma'$ . Hence  $d(x, \text{hull}(Y)) \leq \langle y, y' \rangle_x + 6\delta$ .  $\square$

**Lemma 2.16.** Let  $Y$  and  $Z$  be two subsets of  $X$ . Let  $x$  be a point of  $X$ . Assume that for all  $y \in Y$ , for all  $z \in Z$ ,  $\langle y, z \rangle_x \leq \alpha$ . Then for all  $y \in \text{hull}(Y)$ , for all  $z \in \text{hull}(Z)$ ,  $\langle y, z \rangle_x \leq \alpha + 3\delta$ .

*Proof.* Let  $y \in \text{hull}(Y)$  and  $z \in \text{hull}(Z)$ . By definition there exists  $y_1, y_2 \in Y$  (respectively  $z_1, z_2 \in Z$ ) such that  $y$  (respectively  $z$ ) lies on a  $(1, \delta)$ -quasi-geodesic between  $y_1$  and  $y_2$  (respectively  $z_1$  and  $z_2$ ). By hyperbolicity

$$\min \left\{ \langle y_1, x \rangle_y, \langle y_2, x \rangle_y \right\} \leq \langle y_1, y_2 \rangle_y + \delta \leq \frac{3}{2}\delta.$$

In particular there is  $i \in \{1, 2\}$  such that  $\langle y_i, x \rangle_y \leq 3\delta/2$ . In the same way there is  $j \in \{1, 2\}$  such that  $\langle z_j, x \rangle_z \leq 3\delta/2$ . By triangle inequality we obtain

$$\langle y, z \rangle_x \leq \langle y_i, z_j \rangle_x + \langle y_i, x \rangle_y + \langle z_j, x \rangle_z \leq \alpha + 3\delta. \quad \square$$

## 2.4 Ultra-limit of hyperbolic spaces

Let us first recall the definition of the ultra-limit of a sequence of metric spaces and some related notations.

A non-principal ultra-filter is a finite additive map  $\omega : \mathcal{P}(\mathbf{N}) \rightarrow \{0, 1\}$  such that  $\omega(\mathbf{N}) = 1$  and which vanishes on every finite subset of  $\mathbf{N}$ . A property  $P_n$  is true  $\omega$ -almost surely ( $\omega$ -as) if  $\omega(\{n \in \mathbf{N} \mid P_n \text{ is true}\}) = 1$ . A real sequence  $(u_n)$  is  $\omega$ -essentially bounded ( $\omega$ -eb) if there exists  $M$  such that  $|u_n| \leq M$   $\omega$ -as. Given  $l \in \mathbf{R}$ , we say that the  $\omega$ -limit of  $(u_n)$  is  $l$  and write  $\lim_\omega u_n = l$  if for all  $\varepsilon > 0$ ,  $|u_n - l| \leq \varepsilon$   $\omega$ -as. In particular, any sequence which is  $\omega$ -eb admits a  $\omega$ -limit [2].

Let  $(X_n, x_n^0)$  be a sequence of pointed metric spaces. We define the following set

$$\Pi_\omega X_n = \left\{ (x_n) \in \prod_{n \in \mathbf{N}} X_n \mid (|x_n - x_n^0|) \text{ is } \omega\text{-eb} \right\}.$$

The space  $\Pi_\omega X_n$  is endowed with a pseudo-metric defined in the following way:  $|(x_n) - (y_n)| = \lim_\omega |x_n - y_n|$ .

**Definition 2.17.** The  $\omega$ -limit of  $(X_n, x_n^0)$ , denoted by  $\lim_\omega (X_n, x_n^0)$  or simply  $\lim_\omega X_n$ , is the quotient of the space  $\Pi_\omega X_n$  by the equivalence relation which identifies two points at distance zero. The pseudo-distance on  $\Pi_\omega X_n$  induces a distance on  $\lim_\omega X_n$ .

**Remark.** If the diameter of  $X_n$  is uniformly bounded, then  $\lim_\omega (X_n, x_n^0)$  does not depend on the choice of a base point  $x_n^0$ .

**Notations.**

- Given a sequence  $(x_n) \in \Pi_\omega X_n$  we write  $\lim_\omega x_n$  for its equivalence class in  $\lim_\omega X_n$ .
- For all  $n \in \mathbf{N}$ , let  $Y_n$  be a subset of  $X_n$ . The set  $\lim_\omega Y_n$  is defined by

$$\lim_\omega Y_n = \left\{ \lim_\omega y_n \mid (y_n) \in \Pi_\omega X_n \text{ and } y_n \in Y_n \text{ } \omega\text{-as} \right\}.$$

**Proposition 2.18** (Ultra-limit of hyperbolic spaces [[7, Prop. 1.1.2]]). *Let  $\omega$  be a non-principal ultra-filter. Let  $(X_n, x_n^0)$  be a sequence of pointed  $\delta_n$ -hyperbolic length spaces such that  $\delta = \lim_\omega \delta_n$ . Then  $\lim_\omega X_n$  is a  $\delta$ -hyperbolic geodesic space. In particular, if  $\delta = 0$ ,  $\lim_\omega X_n$  is an  $\mathbf{R}$ -tree.*

**Proposition 2.19** ([7, Prop. 1.1.4]). *Let  $\delta \geq 0$ . Let  $\omega$  be a non-principal ultra-filter. Let  $(X_n, x_n^0)$  be a sequence of pointed length spaces. Assume that  $\lim_\omega X_n$  is a  $\delta$ -hyperbolic space. Then for every  $\eta > 0$  for every  $r > 0$ , every ball of radius  $r$  of  $X_n$  is  $(\delta + \eta)$ -hyperbolic  $\omega$ -as.*

**Proposition 2.20.** *Let  $\omega$  be a non-principal ultra-filter. Let  $(X_n, x_n^0)$  be a sequence of pointed  $\delta_n$ -hyperbolic length spaces with  $\lim_\omega \delta_n = 0$ . For every  $n \in \mathbf{N}$  let  $Y_n$  and  $Z_n$  be respectively  $\alpha_n$ - and  $\beta_n$ -quasi-convex subsets of  $X_n$ . We denote by  $Y = \lim_\omega Y_n$  and  $Z = \lim_\omega Z_n$  the corresponding limit subsets of  $X = \lim_\omega X_n$ . Then*

$$\text{diam}(Y \cap Z) \leq \lim_\omega \text{diam}(Y_n^{+\alpha_n+3\delta_n} \cap Z_n^{+\beta_n+3\delta_n}).$$

*Proof.* Let  $x = \lim_\omega x_n$  and  $x' = \lim_\omega x'_n$  be two points of  $Y \cap Z$ . Let  $A > 0$ . Since  $x$  and  $x'$  belong to both  $Y$  and  $Z$ ,  $x_n$  and  $x'_n$  belong to  $Y_n^{+A} \cap Z_n^{+A}$   $\omega$ -as. Applying Proposition 2.13 we obtain

$$|x'_n - x_n| \leq \text{diam}(Y_n^{+\alpha_n+3\delta_n} \cap Z_n^{+\beta_n+3\delta_n}) + 2A + 4\delta_n \text{ } \omega\text{-as}.$$

After taking the  $\omega$ -limit, it gives

$$|x' - x| \leq \lim_\omega \text{diam}(Y_n^{+\alpha_n+3\delta_n} \cap Z_n^{+\beta_n+3\delta_n}) + 2A.$$

This last inequality holds for every  $x, x' \in Y \cap Z$  and every  $A > 0$ , which leads to the result.  $\square$

## 2.5 Isometries of a hyperbolic space

In this section we assume that the space  $X$  is geodesic and proper. By *proper* we mean that every closed ball of  $X$  is compact. Although it is not necessarily unique,  $[x, x']$  stands for a geodesic between two points  $x$  and  $x'$  of  $X$ . We denote by  $\partial X$  the boundary at infinity of  $X$  (see [6, Chap. 2]). The space  $X$  being proper any two distinct points of  $\partial X$  are joined by a bi-infinite geodesic. In this situation one can precise the constants that appears in Corollary 2.6 (see [4, Chap. III.H, Th. 1.13]): the Hausdorff distance between two  $200\delta$ -local  $(1, 0)$ -quasi-geodesics of  $X$  joining the same extremities (eventually in  $\partial X$ ) is at most  $5\delta$ . In particular the Hausdorff distance between two bi-infinite geodesics joining the same points of  $\partial X$  is at most  $5\delta$ .

**Lemma 2.21.** *Let  $\alpha \geq 0$  and  $Y$  be an  $\alpha$ -quasi-convex subset of  $X$ . For every  $A \geq \alpha + 2\delta$ , the  $A$ -neighborhood of  $Y$  is strongly quasi-convex.*

**Remark.** An analog statement is true if the space  $X$  is not proper of geodesic. However it requires to consider the *open*  $A$ -neighborhood of  $X$ . That is why we preferred to state this result with this stronger assumption.

*Proof.* According to Lemma 2.11, it is sufficient to prove that the  $2\delta$ -neighborhood of a closed  $2\delta$ -quasi-convex subset  $Y$  of  $X$  is  $8\delta$ -strongly quasi-convex. Let  $x$  and  $x'$  be two points of  $X$  which are  $2\delta$ -close to  $Y$ . We denote by  $p$  and  $p'$  their respective projections on  $Y$ . By construction the geodesics  $[x, p]$  and  $[x', p']$  lie in the  $2\delta$ -neighborhood of  $Y$ . By quasi-convexity the same holds for  $[p, p']$ . Thus by concatenating the three geodesics we obtain a path contained in the  $2\delta$ -neighborhood of  $Y$  joining  $x$  to  $x'$  whose length is at most  $|x - x'| + 8\delta$ . Consequently the  $2\delta$ -neighborhood of  $Y$  is  $8\delta$ -strongly quasi-convex.  $\square$

Let  $x$  be a point of  $X$ . An isometry  $g$  of  $X$  is either

- *elliptic* i.e., the orbit of  $x$  under  $g$  is bounded,
- *parabolic* i.e., the orbit of  $x$  under  $g$  has exactly one accumulation point in  $\partial X$ .
- *hyperbolic* i.e., the orbit of  $x$  under  $g$  has exactly two accumulation points in  $\partial X$ .

Note that these definitions do not depend on  $x$ . In order to measure the action of  $g$  on  $X$ , we define two translation lengths. By the *translation length*  $[g]_X$  (or simply  $[g]$ ) we mean

$$[g]_X = \inf_{x \in X} |gx - x|.$$

The *asymptotic translation length*  $[g]_X^\infty$  (or simply  $[g]^\infty$ ) is

$$[g]_X^\infty = \lim_{n \rightarrow +\infty} \frac{1}{n} |g^n x - x|.$$

These two lengths satisfy the following inequality  $[g]^\infty \leq [g] \leq [g]^\infty + 32\delta$  [6, Chap. 10, Prop. 6.4]. An isometry  $g$  of  $X$  is hyperbolic if and only if  $[g]^\infty > 0$  [6, Chap. 10, Prop. 6.3].

**Lemma 2.22.** *Let  $x, x'$  and  $y$  be three points of  $X$ . Let  $g$  be an isometry of  $X$ . Then  $|gy - y| \leq \max\{|gx - x|, |gx' - x'|\} + 2\langle x, x' \rangle_y + 6\delta$ .*

*Proof.* By hyperbolicity

$$\min \left\{ \langle x, gx \rangle_y, \langle gx, gx' \rangle_y, \langle gx', x' \rangle_y \right\} \leq \langle x, x' \rangle_y + 2\delta. \quad (4)$$

Assume that the minimum is achieved by  $\langle x, gx \rangle_y$ . Using the triangle inequality we obtain  $|gy - y| \leq |gx - x| + 2\langle x, gx \rangle_y \leq |gx - x| + 2\langle x, x' \rangle_y + 4\delta$ . A similar inequality holds if the minimum is achieved by  $\langle gx', x' \rangle_y$ . Suppose now that the minimum in (4) is achieved by  $\langle gx, gx' \rangle_y$ . Hence  $\langle gx, gx' \rangle_y \leq \langle x, x' \rangle_y + 2\delta$ . Applying (2) we obtain

$$|gy - y| + |gx - gx'| \leq \max \left\{ |gx - gy| + |gx' - y|, |gx' - gy| + |gx - y| \right\} + 2\delta. \quad (5)$$

However, by triangle inequality

$$\begin{aligned} |gx - gy| + |gx' - y| &\leq |gx - x| + |gx' - gx| + 2\langle gx, gx' \rangle_y \\ &\leq |gx - x| + |gx' - gx| + 2\langle x, x' \rangle_y + 4\delta. \end{aligned}$$

The same inequality holds after swapping  $x$  and  $x'$ . Therefore (5) leads to the desired result.  $\square$

**Definition 2.23.** Let  $g$  be an isometry of  $X$ . The *axis* of  $g$  denoted by  $A_g$  is the set of points  $x \in X$  such that  $|gx - x| \leq \max\{[g], 8\delta\}$ .

**Remarks.** Note that we do not require  $g$  to be hyperbolic. This definition works also for parabolic or elliptic isometries. This subset is not empty because  $X$  is proper. It is also closed.

**Proposition 2.24.** Let  $g$  be an isometry of  $X$ . Let  $x$  be a point of  $X$ .

- (i)  $|gx - x| \geq 2d(x, A_g) + [g] - 14\delta$ ,
- (ii) if  $|gx - x| \leq [g] + A$ , then  $d(x, A_g) \leq \frac{1}{2}A + 7\delta$ ,
- (iii)  $A_g$  is  $14\delta$ -quasi-convex.

*Proof.* Let  $x \in X$ . Note that if  $x$  belongs to  $A_g$ , Point (i) is true. Therefore we can assume that  $x \notin A_g$ . We denote by  $y$  a projection of  $x$  on  $A_g$ . Observe that any geodesic  $[y, gy]$  is contained in  $A_g$ . Such a geodesic is  $3\delta$ -quasi-convex. Moreover,  $y$  and  $gy$  are respective projections of  $x$  and  $gx$  on it. Proposition 2.10 gives

$$|gy - y| \leq \max\{|gx - x| - 2|x - y| + 14\delta, 7\delta\}. \quad (6)$$

On the other hand we claim that  $|gy - y| \geq 8\delta$ . Recall that  $x$  does not belong to  $A_g$ . By construction of  $y$  any point  $z$  on  $[x, y]$  distinct from  $y$  does not belong to  $A_g$ . Therefore  $|gz - z| \geq \max\{[g], 8\delta\}$ . Taking the limit as  $z$  approaches  $y$  leads to the claim. Hence (6) gives

$$|gx - x| \geq |gy - y| + 2|x - y| - 14\delta \geq [g] + 2d(x, A_g) - 14\delta, \quad (7)$$

which proves Point (i). Point (ii) is a consequence of (i). Let us now prove Point (iii). Let  $y$  and  $y'$  be two points of  $A_g$ . Let  $x$  be a point of  $X$ . By Lemma 2.22,

$$|gx - x| \leq \max\{|gy - y|, |gy' - y'|\} + 2\langle y, y' \rangle_x + 6\delta \leq [g] + 2\langle y, y' \rangle_x + 14\delta.$$

It follows then from Point (ii) that  $d(x, A_g) \leq \langle y, y' \rangle_x + 14\delta$ .  $\square$

Let  $g$  be a hyperbolic isometry of  $X$ . We write  $g^-$  and  $g^+$  for the accumulation points in  $\partial X$  of an orbit of  $g$ . They are the only points of  $\partial X$  fixed by  $g$ .

**Definition 2.25.** Let  $g$  be a hyperbolic isometry of  $X$ . The *cylinder* of  $g$  denoted by  $Y_g$  is the set of points lying in the  $10\delta$ -neighborhood of a geodesic joining  $g^-$  and  $g^+$ .

**Lemma 2.26.** Let  $g$  be a hyperbolic isometry of  $X$ . The union  $\Gamma$  of all geodesics joining  $g^-$  to  $g^+$  is  $8\delta$ -quasi-convex. The set  $Y_g$  is strongly quasi-convex.

*Proof.* According to Lemma 2.21 it is sufficient to prove that  $\Gamma$  is  $8\delta$ -quasi-convex. Let  $x \in X$  and  $y, y' \in \Gamma$ . There exist  $\gamma$  and  $\gamma'$  two geodesics joining  $g^-$  and  $g^+$  such that  $y$  and  $y'$  respectively lie on  $\gamma$  and  $\gamma'$ . We denote by  $p'$  a projection of  $y'$  on  $\gamma$ . Since  $\gamma$  and  $\gamma'$  join the same extremities the Hausdorff distance between them is at most  $5\delta$ . Thus  $|y' - p'| \leq 5\delta$ . The path  $\gamma$  being a bi-infinite geodesic, it follows that

$$d(x, \Gamma) \leq d(x, \gamma) \leq \langle y, p' \rangle_x + 3\delta \leq \langle y, y' \rangle_x + 8\delta. \quad \square$$

**Lemma 2.27.** *Let  $g$  be a hyperbolic isometry of  $X$ . Let  $Y$  be a  $g$ -invariant  $\alpha$ -quasi-convex subset of  $X$ . Then  $Y_g$  lies in the  $(\alpha + 22\delta)$ -neighborhood of  $Y$ . In particular  $Y_g$  is contained in the  $36\delta$ -neighborhood of  $A_g$ .*

*Proof.* It is sufficient to prove that every bi-infinite geodesic joining  $g^-$  to  $g^+$  lies in the  $(\alpha + 12\delta)$ -neighborhood of  $Y$ . Let  $\gamma$  be such a geodesic and  $x$  a point of  $\gamma$ . Let  $\eta > 0$ . We denote by  $y$  an  $\eta$ -projection of  $x$  on  $Y$ . Since  $g$  is hyperbolic there exists  $m \in \mathbf{N}$  such that  $|g^m x - g^{-m} x| > 2|x - y| + 26\delta$ . The geodesics  $g^m \gamma$  and  $g^{-m} \gamma$  also join  $g^-$  and  $g^+$ . Therefore  $g^m x$  and  $g^{-m} x$  are  $5\delta$ -close to  $\gamma$ . We denote by  $p_-$  and  $p_+$  respective projections of these points on  $\gamma$ . Note that  $x$  lies on the portion of  $\gamma$  between  $p_-$  and  $p_+$ . Indeed if it was not the case we would have

$$|g^m x - g^{-m} x| \leq |p_- - p_+| + 10\delta = \left| |x - p_-| - |x - p_+| \right| + 10\delta \leq 20\delta,$$

which contradicts our assumption on  $m$ . In particular  $\langle g^m x, g^{-m} x \rangle_x \leq 10\delta$ . By hyperbolicity we get

$$\min \{ |g^m x - x| - |x - y|, \langle g^m y, g^{-m} y \rangle_x \} \leq \langle g^m x, g^{-m} x \rangle_x + 2\delta \leq 12\delta$$

By construction of  $m$ ,  $|g^m x - x|$  is bounded below by  $|x - y| + 13\delta$ . Therefore the minimum in the previous inequality is achieved by  $\langle g^m y, g^{-m} y \rangle_x$ . However  $Y$  being  $g$ -invariant and  $\alpha$ -quasi-convex,  $g^m y$  and  $g^{-m} y$  are two points of  $Y$  and  $d(x, Y) \leq \langle g^m y, g^{-m} y \rangle_x + \alpha$ . Consequently  $x$  lies in the  $(\alpha + 12\delta)$ -neighborhood of  $Y$ .  $\square$

Let  $g$  be an isometry of  $X$  such that  $[g] > 200\delta$ . (In particular,  $g$  is hyperbolic.) Let  $x$  be a point of  $A_g$ . We consider a geodesic  $\gamma : J \rightarrow X$  between  $x$  and  $gx$  parametrized by arc length. We extend  $\gamma$  in a  $g$ -invariant path  $\gamma : \mathbf{R} \rightarrow X$  in the following way: for all  $t \in J$ , for all  $m \in \mathbf{Z}$ ,  $\gamma(t + m[g]) = g^m \gamma(t)$ . This is a  $[g]$ -local  $(1, 0)$ -quasi-geodesic contained in  $A_g$  joining  $g^-$  to  $g^+$ . By stability of quasi-geodesics  $\gamma$  is actually  $8\delta$ -quasi-convex. We call such a path a *nerve* of  $g$ . The Hausdorff distance between two nerves of  $g$  is at most  $5\delta$ .

**Lemma 2.28.** *Let  $g \in G$  such that  $[g] > 200\delta$ . Let  $\gamma$  be a  $200\delta$ -local  $(1, 0)$ -quasi-geodesic joining  $g^-$  to  $g^+$ . Then  $A_g$  is contained in the  $5\delta$ -neighborhood of  $\gamma$ . In particular  $A_g$  is contained in  $Y_g$  and in the  $5\delta$ -neighborhood of any nerve of  $g$ .*

*Proof.* Let  $x \in A_g$ . There exists  $\gamma'$  a nerve of  $g$  going through  $x$ . Both  $\gamma$  and  $\gamma'$  are  $200\delta$ -local  $(1, 0)$ -quasi-geodesic joining  $g^-$  to  $g^+$ . By the stability of quasi-geodesics  $\gamma'$  and thus  $x$  lies in the  $5\delta$ -neighborhood of  $\gamma$ .  $\square$

The next lemma explains the following fact. Let  $g$  be a hyperbolic isometry of  $X$ . A quasi-geodesic contained in the neighborhood of the axis of  $g$  almost behaves like a nerve of  $g$ .

**Lemma 2.29.** *Let  $g \in G$  such that  $[g] > 200\delta$ . Let  $\gamma : [a, b] \rightarrow X$  be a  $[g]$ -local  $(1, 0)$ -quasi-geodesic contained in the  $A$ -neighborhood of  $A_g$ . Then there exists  $\varepsilon \in \{\pm 1\}$  such that for every  $s \in [a, b]$  if  $s \leq b - [g]$  then*

$$|g^\varepsilon \gamma(s) - \gamma(s + [g])| \leq 4A + 80\delta.$$

*Proof.* We denote by  $\gamma_g$  an nerve of  $g$ . Since  $[g] > 200\delta$ , the  $5\delta$ -neighborhood of  $\gamma_g$  contains  $A_g$ . Thus  $\gamma$  lies in the  $(A + 5\delta)$ -neighborhood of  $\gamma_g$ . In particular there exist  $c, d \in \mathbf{R}$  such that  $|\gamma(a) - \gamma_g(c)| \leq A + 5\delta$  and  $|\gamma(b) - \gamma_g(d)| \leq A + 5\delta$ . By replacing if necessary  $g$  by  $g^{-1}$  we can assume that  $c \leq d$ .



Let  $s \in [a, b]$  such that  $s \leq t - [g]$ . Using the stability of quasi-geodesics there exist  $t \in [c, d]$  and  $t' \in [t, d]$  such that  $|\gamma(s) - \gamma_g(t)| \leq A + 15\delta$  and  $|\gamma(s + [g]) - \gamma_g(t')| \leq A + 19\delta$ . It follows that

$$\left| |\gamma_g(t) - \gamma_g(t')| - |\gamma(s) - \gamma(s + [g])| \right| \leq 2A + 34\delta.$$

However  $|\gamma(s) - \gamma(s + [g])|$  and  $|\gamma_g(t) - \gamma_g(t + [g])|$  both equal  $[g]$ . Consequently

$$\left| |\gamma_g(t) - \gamma_g(t')| - |\gamma_g(t) - \gamma_g(t + [g])| \right| \leq 2A + 34\delta.$$

Since  $t'$  and  $t + [g]$  are larger than  $t$  we get by Lemma 2.2-(ii).

$$|\gamma_g(t') - \gamma_g(t)| = |\gamma_g(t') - \gamma_g(t + [g])| \leq 2A + 46\delta.$$

It follows then from the triangle inequality that  $|\gamma(s) - \gamma(s + [g])| \leq 4A + 80\delta$ .  $\square$

## 2.6 Group acting on a hyperbolic space

In this section  $G$  denotes a group acting by isometries on  $X$ . We still assume that  $X$  is geodesic and proper. Moreover, we require the action of  $G$  on  $X$  to be

- (i) *proper* i.e., for every  $x \in X$ , there exists  $r > 0$  such that the set of elements  $g \in G$  satisfying  $gB(x, r) \cap B(x, r) \neq \emptyset$  is finite.
- (ii) *co-compact* i.e., the quotient  $X/G$  endowed with the induced topology is compact.

Since the space  $X$  is proper, the properness of the action of  $G$  implies this more general fact. Let  $Y$  be a bounded subset of  $X$ . The set of elements  $g \in G$  such that  $gY$  intersects  $Y$  is finite [4, Chap. I.8, Remark 8.3]. It follows from these assumptions that a subgroup of  $G$  is either *elementary* i.e., virtually cyclic or contains a free group of rank 2 [11, Chap. 8, Th. 37].

**Notation.** Given a subset  $Y$  of  $X$  we denote by  $\text{Stab}(Y)$  the stabilizer of  $Y$  i.e., the set of elements  $g \in G$  such that  $gY = Y$ .

**Finite subgroups.** We start by studying some properties of the finite subgroups of  $G$ . To that end we associate to each such subgroup a particular subset of  $X$ .

**Definition 2.30.** Given a finite subgroup  $F$  of  $G$  we denote by  $C_F$  the set of points  $x \in X$  such that for every  $g \in F$ ,  $|gx - x| \leq 10\delta$ .

It follows from the definition that  $C_F$  is an  $F$ -invariant subset of  $X$ .

**Proposition 2.31.** *Let  $F$  be a finite subgroup of  $G$ . Let  $x$  be a point of  $X$ . Let  $g \in F$  such that  $|gx - x|$  is maximal. We denote by  $m$  the midpoint of a geodesic  $[x, gx]$ . Then  $m$  belongs to  $C_F$ . In particular  $C_F$  is non-empty.*

*Proof.* Let  $h$  be an element of  $F$ . In order to simplify the notations we put  $y = gx$ ,  $z = hx$  and  $t = hgx$ . The point  $p = hm$  is thus the midpoint of  $[z, t]$ . The only fact that we are going to use is that  $|x - y| = |z - t|$  is the largest distance between any two points of  $\{x, y, z, t\}$ . Using hyperbolicity condition (2) we have

$$|x - y| + |z - t| \leq \max\{|x - t| + |y - z|, |y - t| + |x - z|\} + 2\delta. \quad (8)$$

Note that  $z$  and  $t$  play a symmetric role. Without loss of generality we can assume that the maximum is achieved by  $|x - t| + |y - z|$ . It follows that

$$\left(|x - y| - |x - t|\right) + \left(|z - t| - |y - z|\right) \leq 2\delta.$$

Recall that  $|x - y|$  and  $|z - t|$  are respectively larger than or equal to  $|x - t|$  and  $|y - z|$ . Consequently  $0 \leq |x - y| - |x - t| \leq 2\delta$ . Roughly speaking the triangle  $[x, y, t]$  has two sides with approximatively the same length namely  $[x, y]$  and  $[x, t]$ , this length being larger than the one of the last side. It follows that  $\langle y, t \rangle_x \geq |x - m| - \delta$ . Similarly we have  $\langle x, z \rangle_t \geq |t - p| - \delta$ . Applying twice Lemma 2.2-(i), we obtain

$$\begin{aligned} \langle x, t \rangle_m &\leq \max\left\{|x - m| - \langle y, t \rangle_x, \langle x, y \rangle_m\right\} + \delta \leq 2\delta, \\ \langle x, t \rangle_p &\leq \max\left\{|t - p| - \langle x, z \rangle_t, \langle z, t \rangle_p\right\} + \delta \leq 2\delta. \end{aligned}$$

Lemma 2.2-(ii), leads then to

$$|p - m| \leq \left||x - p| - |x - m|\right| + 2 \max\left\{\langle x, t \rangle_m, \langle x, t \rangle_p\right\} + 2\delta \leq \left||x - p| - |x - m|\right| + 6\delta.$$

However  $m$  and  $p$  are respectively the midpoints of  $[x, y]$  and  $[z, t]$  which have the same length, thus

$$|x - p| - |x - m| = \left(|x - t| - |x - y|\right) + 2\langle x, t \rangle_p.$$

Therefore  $|p - m| \leq 10\delta$  i.e.,  $|hm - m| \leq 10\delta$ . In other words  $m$  belongs to  $C_F$ .  $\square$

**Corollary 2.32.** *Let  $F$  be a finite subgroup of  $G$ . The subset  $C_F$  is  $8\delta$ -quasi-convex.*

*Proof.* Let  $x \in X$  and  $y, y' \in C_F$ . We denote by  $g$  an element of  $F$  such that  $|gx - x|$  is maximal and  $m$  the midpoint of  $[x, gx]$ . According to Lemma 2.22,

$$2|x - m| = |gx - x| \leq \max\{|gy - y|, |gy' - y'|\} + 2\langle y, y' \rangle_x + 6\delta.$$

However  $y$  and  $y'$  belong to  $C_F$ , hence  $|x - m| \leq \langle y, y' \rangle_x + 8\delta$ . By Proposition 2.31,  $m$  belongs to  $C_F$ . Therefore  $d(x, C_F) \leq \langle y, y' \rangle_x + 8\delta$   $\square$

**Corollary 2.33.** *Let  $F$  be a finite subgroup of  $G$ . Let  $Y$  be a non-empty,  $F$ -invariant,  $\alpha$ -quasi-convex subset of  $X$ . Then the  $\alpha$ -neighborhood of  $Y$  intersects  $C_F$ .*

*Proof.* Let  $x$  be a point of  $Y$ . We denote by  $g$  an element of  $F$  such that  $|gx - x|$  is maximal and  $m$  the midpoint of  $[x, gx]$ . According to Proposition 2.31,  $m$  lies in  $C_F$ . On the other hand,  $Y$  is  $F$ -invariant, therefore  $gx \in Y$ . Since  $Y$  is  $\alpha$ -quasi-convex,  $d(y, Y) \leq \langle x, gx \rangle_m + \alpha \leq \alpha$ . Consequently  $m$  belongs to the  $\alpha$ -neighborhood of  $Y$ .  $\square$

### Infinite elementary subgroups.

Let  $H$  be an infinite elementary subgroup of  $G$ . By definition  $H$  contains a finite index subgroup isomorphic to  $\mathbf{Z}$ . The set of accumulation points in  $\partial X$  of an orbit of  $H$ , that we denote  $\partial H$ , has exactly two points. There exists a subgroup  $H^+$  of  $H$  of index at most 2 which fixes pointwise  $\partial H$ . If  $H^+ \neq H$  then  $H$  contains an element of order 2. A Schur Theorem (see [28, Th. 5.32]) implies that  $H^+$  contains a unique maximal finite subgroup  $F$ . This group is actually a normal subgroup of  $H^+$ . Moreover, there exists a hyperbolic element  $h \in H^+$  so that  $H^+$  is isomorphic to  $F \rtimes \mathbf{Z}$  where  $\mathbf{Z}$  is identified with the subgroup  $\langle h \rangle$  acting by conjugation on  $F$ .

Let  $g$  be a hyperbolic element of  $G$ . The subgroup  $E$  of  $G$  that stabilizes  $\{g^-, g^+\}$  is the maximal elementary subgroup of  $G$  containing  $g$  [6, Chap. 10, Prop. 7.1]. The isometry  $g$  is said to be a *proper power* if there exist  $h \in G$  and an integer  $n \geq 2$  such that  $g = h^n$ . Any hyperbolic element of  $G$  is a power of an isometry which is not a proper power.

**Lemma 2.34.** *Let  $g$  be a hyperbolic element of  $G$  and  $H$  a subgroup of  $G$  fixing pointwise  $\{g^-, g^+\}$ . Let  $F$  be the maximal finite subgroup of  $H$ . The cylinder  $Y_g$  of  $g$  is contained in the  $48\delta$ -neighborhood of  $C_F$ .*

*Proof.* According to Lemma 2.26, the union  $\Gamma$  of all geodesics joining  $g^-$  to  $g^+$  is  $8\delta$ -quasi-convex. It is also  $F$ -invariant. By Lemma 2.33, there exists a point  $x$  in  $C_F \cap Y_g$ . The subgroup  $F$  being  $g$  invariant, for every  $n \in \mathbf{Z}$ ,  $g^n x$  belongs to  $C_F \cap Y_g$ . Note that  $g^n x$  tends to  $g^+$  (respectively  $g^-$ ) and  $n$  approaches  $+\infty$  (respectively  $-\infty$ ). In particular, for every  $y \in Y_g$  there exist  $n, m \in \mathbf{Z}$  such that  $\langle g^n x, g^m x \rangle_y \leq 40\delta$ . Since  $C_F$  is  $8\delta$ -quasi-convex,  $y$  lies in the  $48\delta$ -neighborhood of  $C_F$ .  $\square$

**Lemma 2.35.** *Assume that every elementary subgroup of  $G$  is cyclic. Let  $g, h \in G$ . If  $g$  and  $hgh^{-1}$  generate an elementary subgroup then so do  $g$  and  $h$ .*

*Proof.* We denote by  $H$  the subgroup of  $G$  generated by  $g$  and  $hgh^{-1}$ . We distinguish two cases. If  $g$  is hyperbolic then  $g$  and  $hgh^{-1}$  are two hyperbolic isometries with the same accumulation points in  $\partial X$ . In particular  $h$  stabilizes the set  $\{g^-, g^+\}$ . It follows that  $g$  and  $h$  belong to the elementary subgroup  $E(g)$ .

Assume now that  $g$  has finite order. The subgroup  $H$  is elementary, thus cyclic. In particular it has to be finite. The isometries  $g$  and  $hgh^{-1}$  generate two subgroups of  $H$  with the same order. However two such subgroups in a cyclic group are equal. Therefore there exists  $m \in \mathbf{Z}$  such that  $hgh^{-1} = g^m$ . Thus any element of the subgroup generated by  $g$  and  $h$  can be written  $g^p h^q$  with  $p, q \in \mathbf{Z}$ . Since  $g$  has finite order, this subgroup is also elementary.  $\square$

**Lemma 2.36.** *We assume that every elementary subgroup of  $G$  is cyclic. Let  $n \in \mathbf{N}$ . Let  $g$  and  $h$  be two hyperbolic elements of  $G$  which are not proper powers. Either  $g$  and  $h$  generate a non-elementary subgroup of  $G$  or  $\langle g^n \rangle = \langle h^n \rangle$ .*

*Proof.* Assume that  $g$  and  $h$  generate an elementary subgroup. This subgroup is infinite and cyclic. Since  $g$  and  $h$  are not proper powers, they are either equal or inverse. Hence  $\langle g^n \rangle = \langle h^n \rangle$ .  $\square$

**Group invariants.** We now introduce several invariants associated to the action of  $G$  on  $X$ . During the final induction, they will be useful to ensure that the set of relations we are looking at satisfy a small cancellation assumption.

**Definition 2.37.** Let  $P$  be a subset of  $G$ . The *injectivity radius* of  $P$  on  $X$ , denoted by  $r_{inj}(P, X)$  is

$$r_{inj}(P, X) = \inf \{[g]^\infty \mid g \in P, \text{ hyperbolic}\}$$

**Proposition 2.38** (see [9, Prop. 3.1]). *There exists  $a > 0$  such that for every hyperbolic element  $g \in G$  we have  $[g]^\infty \in a\mathbf{N}$ . In particular  $r_{inj}(G, X) > 0$ .*

**Definition 2.39.** We denote by  $\mathcal{A}$  the set of pairs  $(g, h)$  generating a non-elementary subgroup of  $G$  such that  $[g] \leq 1000\delta$  and  $[h] \leq 1000\delta$ . The parameter  $A(G, X)$  is given by

$$A(G, X) = \sup_{(g, h) \in \mathcal{A}} \text{diam} (A_g^{+17\delta} \cap A_h^{+17\delta})$$

The invariant  $A(G, X)$  depends implicitly on the hyperbolicity constant  $\delta$ . Although the notation does not make this dependency explicit, we should keep in mind that it plays an important role. For instance, we have the following lemma:

**Lemma 2.40.** *Let  $\lambda$  be a positive number. We denote by  $\lambda X$  the space  $X$  endowed with the rescaled metric  $\lambda \mid \cdot \mid_X$  and view it as a  $\lambda\delta$ -hyperbolic space. Then  $A(G, \lambda X) = \lambda A(G, X)$ .*

*Proof.* Let  $g$  be an element of  $G$ . Its translation length satisfies  $[g]_{\lambda X} = \lambda[g]_X$ . Since  $\lambda X$  is a  $\lambda\delta$ -hyperbolic space, the axis of  $g$  in  $\lambda X$  is exactly the image in  $\lambda X$  of the axis  $A_g$  of  $g$  in  $X$ . We will denote it by  $\lambda A_g$ . Let  $g$  and  $g'$  be two elements of  $G$  that do not generate an elementary subgroup and whose translation lengths in  $\lambda X$  are at most  $1000\lambda\delta$ . In particular, we have  $[g]_X, [g']_X \leq 1000\delta$ . By definition of  $A(G, X)$ , we get

$$\begin{aligned} \text{diam} (\lambda A_g^{+17\lambda\delta} \cap \lambda A_{g'}^{+17\lambda\delta}) &= \lambda \text{diam} (A_g^{+17\delta} \cap A_{g'}^{+17\delta}) \\ &\leq \lambda A(G, X). \end{aligned}$$

After taking the upper bound for all  $g$  and  $g'$ , we obtain  $A(G, \lambda X) \leq \lambda A(G, X)$ . In the same way,  $A(G, \lambda X) \geq \lambda A(G, X)$ . This establishes the desired equality.  $\square$

**Proposition 2.41.** *We assume that every elementary subgroup of  $G$  is cyclic. Let  $g$  and  $h$  be two elements of  $G$  which generate a non-elementary subgroup.*

(i) *If  $[g] \leq 1000\delta$ , then  $\text{diam} (A_g^{+17\delta} \cap A_h^{+17\delta}) \leq [h] + A(G, X) + 158\delta$ .*

(ii) *Without assumption on  $g$  we have,*

$$\text{diam} (A_g^{+17\delta} \cap A_h^{+17\delta}) \leq [g] + [h] + \max\{[g], [h]\} + A(G, X) + 676\delta.$$

*Proof.* We prove Point (i) by contradiction. Assume that

$$\text{diam} (A_g^{+17\delta} \cap A_h^{+17\delta}) > [h] + A(G, X) + 158\delta.$$

By definition of  $A(G, X)$  we have  $[h] > 1000\delta$ , otherwise  $g$  and  $h$  would generate an elementary subgroup. We denote by  $\gamma : \mathbf{R} \rightarrow X$  a nerve of  $h$ . Its  $5\delta$ -neighborhood contains  $A_g$  (Lemma 2.28) therefore by Proposition 2.13

$$\text{diam}(A_g^{+17\delta} \cap \gamma^{+11\delta}) > [h] + A(G, X) + 110\delta.$$

In particular there exist two points on  $\gamma$ ,  $x = \gamma(s)$  and  $x' = \gamma(s')$  which also belong to the  $28\delta$ -neighborhood of  $A_g$  and such that

$$|x - x'| > [h] + A(G, X) + 88\delta. \quad (9)$$

By replacing if necessary  $h$  by  $h^{-1}$  we can assume that  $s \leq s'$ . By stability of quasi-geodesics for all  $t \in [s, s']$ ,  $\langle x, x' \rangle_{\gamma(t)} \leq 5\delta$ . Since the  $28\delta$ -neighborhood of  $A_g$  is  $2\delta$ -quasi-convex (see Lemma 2.11),  $\gamma(t)$  lies in the  $35\delta$ -neighborhood of  $A_g$ . Thus  $|g\gamma(t) - \gamma(t)| \leq [g] + 70\delta$ . According to (9) there exists  $t \in [s, s']$  such that  $|x' - \gamma(t)| = [h]$ . We put  $y = \gamma(t)$ . By construction  $hx = \gamma(s + [h])$  and  $hy = \gamma(t + [h])$ . Note that  $|s' - t| \geq [h]$ , thus  $s + [h]$  and  $t + [h]$  belong to  $[s, s']$ . Hence

$$|ghx - hx|, |ghy - hy| \leq [hgh^{-1}] + 70\delta.$$

It follows from Proposition 2.24, that  $x$  and  $y$  belong to the  $42\delta$ -neighborhood of  $hA_g$ . Consequently  $x$  and  $y$  are two points of  $A_g^{+35\delta} \cap hA_g^{+42\delta}$ . By Proposition 2.13,

$$\text{diam}(A_g^{+17\delta} \cap A_{hgh^{-1}}^{+17\delta}) \geq |x - y| - 88\delta \geq |x' - x| - |x' - y| - 88\delta > A(G, X).$$

Moreover,  $[hgh^{-1}] = [g] \leq 1000\delta$ . By definition of  $A(G, X)$  the isometries  $g$  and  $hgh^{-1}$  generate an elementary group. It follows from Lemma 2.35 that  $g$  and  $h$  also generate an elementary group. Contradiction.

We now prove Point (ii). According to the previous point we can assume that  $[g] > 1000\delta$  and  $[h] > 1000\delta$ . Without loss of generality we can suppose  $[h] \geq [g]$ . Imagine now that

$$\text{diam}(A_g^{+17\delta} \cap A_h^{+17\delta}) > [g] + 2[h] + A(G, X) + 676\delta.$$

We denote by  $\gamma$  a nerve of  $h$ . Its  $5\delta$ -neighborhood contains  $A_h$  thus

$$\text{diam}(\gamma^{+11\delta} \cap A_g^{+17\delta}) > [g] + 2[h] + A(G, X) + 628\delta.$$

In particular there exist  $x = \gamma(s)$ ,  $x' = \gamma(s')$  lying in the  $28\delta$ -neighborhood of  $A_g$  such that

$$|x - x'| > [g] + 2[h] + A(G, X) + 616\delta.$$

Without loss of generality we can assume that  $s \leq s'$ . As previously, the restriction of  $\gamma$  to  $[s, s']$  is contained in the  $35\delta$ -neighborhood of  $A_g$ . We apply Lemma 2.29. By replacing if necessary  $g$  by  $g^{-1}$ , for every  $t \in [s, s']$  if  $t \leq s' - [g]$  then  $|g\gamma(t) - \gamma(t + [g])| \leq 220\delta$ . Consequently, for every  $t \in [s, s']$  such that  $t \leq s' - [g] - [h]$  we have

$$|gh\gamma(t) - hg\gamma(t)| \leq |g\gamma(t + [h]) - h\gamma(t + [g])| + 220\delta \leq 440\delta.$$

It follows that the translation length of the isometry  $u = h^{-1}g^{-1}hg$  is at most  $1000\delta$  and for all  $t \in [s, s']$  if  $t \leq s' - [g] - [h]$  then  $\gamma(t)$  is in the  $227\delta$ -neighborhood of  $A_u$  (Proposition 2.24). Let

$y = \gamma(t)$  be the point of  $\gamma$  such that  $|x' - y| = [h] + [g]$ . In particular  $x$  and  $y$  belong to the  $227\delta$ -neighborhood of  $A_u$  and  $A_h$ . Therefore

$$\text{diam}(A_u^{+17\delta} \cap A_h^{+17\delta}) \geq |x - y| - 458\delta \geq |x' - x| - |x' - y| - 458\delta > [h] + A(G, X) + 158\delta.$$

It follows from the previous point that  $h$  and  $u$  generate an elementary subgroup. Hence so do  $h$  and  $g^{-1}hg$ . However  $h$  is a hyperbolic isometry. Consequently  $g$  and  $h$  generate an elementary group. Contradiction.  $\square$

### 3 Rotation families

In this section we follow the presentation of rotation family given by F. Dahmani, V. Guirardel and D. Osin in [8]. For our purpose we only need a weaker statement about rotation family. On the other hand, we work with an eventually non-geodesic space. In order to have a self contained text, we recall the main ideas of the proofs.

Let  $G$  be a group acting by isometries on a  $\delta$ -hyperbolic length space  $X$ . Note that in this section we do not require the space  $X$  to be geodesic or proper. Similarly there is no assumption on the action of  $G$  on  $X$ .

**Definition 3.1.** Let  $\sigma > 0$ . A  $\sigma$ -rotation family is a collection  $\mathcal{R}$  of pairs  $(H, v)$  where  $H$  is a subgroup of  $G$  and  $v$  a point of  $X$  satisfying the following properties.

- (R1) For every  $(H, v) \in \mathcal{R}$ ,  $H$  is subgroup of  $\text{Stab}(v)$  such that for every  $x \in B(v, \sigma/10)$ , for every  $h \in H \setminus \{1\}$ ,  $|hx - x| = 2|v - x|$ .
- (R2) For every  $(H, v), (H', v') \in \mathcal{R}$ , if  $(H, v) \neq (H', v')$ , then  $|v - v'| \geq \sigma$ .
- (R3)  $\mathcal{R}$  is stable under the action of  $G$  defined as follows. For all  $g \in G$ , for all  $(H, v) \in \mathcal{R}$ ,  $g.(H, v) = (gHg^{-1}, gv)$ .

**Remark.** It follows from (R2) and (R3) that for every  $(H, v) \in \mathcal{R}$ ,  $H$  is actually a normal subgroup of  $\text{Stab}(v)$ .

**Notations.** Let  $(H, v) \in \mathcal{R}$ . The idea is that each element  $h \in H$  acts on  $X$  like a rotation of center  $v$  and very large angle - see Axiom (R1). Therefore  $v$  is called an *apex* and  $H$  a *rotation group*. We denote by  $v(\mathcal{R})$  the set of all apices. Similarly  $H(\mathcal{R})$  stands for the set of all rotation groups  $H$ . Given a subset  $Y$  of  $X$ , we denote by  $K_Y$  the subgroup of  $G$  generated by all  $H$ 's, where  $(H, v) \in \mathcal{R}$  and  $v \in Y$ .

#### 3.1 Fundamental theorem

**Theorem 3.2.** There exists a positive number  $\sigma_0 = \sigma_0(\delta)$  depending only on  $\delta$  with the following property. Let  $\mathcal{R}$  be a  $\sigma$ -rotation family with  $\sigma \geq \sigma_0$  and  $K$  the (normal) subgroup of  $G$  generated by all rotation groups  $H \in H(\mathcal{R})$ . Let  $x \in X$  and  $g \in K$ . If  $g$  does not belong to any of the rotation groups  $H \in H(\mathcal{R})$  then  $|gx - x| \geq \sigma - 166\delta$ .

The rest of this section is dedicated to the proof of the theorem. We need first to define  $\sigma_0$ . According to Propositions 2.5 and 2.7 there exists  $\sigma_0 > 0$  depending only on  $\delta$  with the following properties.

- For every  $l \in [0, \delta]$  any  $\sigma_0/25$ -local  $(1, l)$ -quasi-geodesic is a (global)  $(2, l)$ -quasi-geodesic.
- Assume that  $y_0, \dots, y_{m+1}$  is a sequence of points of  $X$  such that for every  $i \in \{1, \dots, m-1\}$ ,  $|y_{i+1} - y_i| \geq \sigma_0$  and for every  $i \in \{1, \dots, m\}$ ,  $\langle y_{i+1}, y_{i-1} \rangle_{y_i} \leq 7\delta$  then for every  $i \in \{0, \dots, m+1\}$ ,  $\langle y_0, y_{m+1} \rangle_{y_i} \leq 12\delta$ . Moreover, for all  $x \in X$  there exists  $i \in \{0, \dots, m\}$ , such that  $\langle y_{i+1}, y_i \rangle_x \leq \langle y_0, y_{m+1} \rangle_x + 22\delta$ .

Without loss of generality we can require that  $\sigma_0 \geq 10^{20}\delta$ . From now on we assume that  $\mathcal{R}$  is a  $\sigma$ -rotation family with  $\sigma \geq \sigma_0$ .

**Lemma 3.3.** *Let  $(H, v) \in \mathcal{R}$ . Let  $h \in H \setminus \{1\}$ . For every  $x \in X$ ,  $\langle x, hx \rangle_v \leq 2\delta$ .*

*Proof.* Let  $x \in X$ . If  $|x - v| < \sigma/10$  the lemma is just a consequence of (R1). Hence we can assume that  $|x - v| \geq \sigma/10$ . We denote by  $y$  a point of  $X$  such that  $|v - y| = \sigma/20$  and  $\langle x, v \rangle_y \leq \delta$ . By hyperbolicity,

$$\min \{ \langle hy, hx \rangle_v, \langle hx, x \rangle_v, \langle x, y \rangle_v \} \leq \langle hy, y \rangle_v + 2\delta = 2\delta.$$

However  $\langle x, y \rangle_v = |v - y| - \langle x, v \rangle_y > 2\delta$ . Therefore the minimum cannot be achieved by  $\langle x, y \rangle_v$  or  $\langle hx, hy \rangle_v$ . Consequently  $\langle hx, x \rangle_v \leq 2\delta$ .  $\square$

**Definition 3.4.** A non-empty subset  $W$  is *windmill* if it satisfies the following conditions.

- (W1)  $W$  is  $2\delta$ -quasi-convex,
- (W2)  $W$  is stable under the action of  $K_W$ ,
- (W3) for every  $v \in v(\mathcal{R})$ , if  $d(v, W) \leq \sigma/10$ , then  $v$  belongs to  $W$ ,
- (W4) for every  $g \in K_W$ , for every  $x \in X$ , if  $g$  does not belong to any rotation group  $H \in H(\mathcal{R})$  then  $|gx - x| \geq \sigma - 166\delta$ .

**Proposition 3.5.** *Let  $W$  be a windmill. There exists a windmill  $W'$  which contains the  $\sigma/10$ -neighborhood of  $W$ .*

*Proof.* Let us denote by  $V$  the following set of apices

$$V = \left\{ v \in v(\mathcal{R}) \setminus W \mid d(v, W) \leq 3\sigma/10 \right\}.$$

If  $V$  is empty then the  $\sigma/10$ -neighborhood of  $W$  is also a windmill. Therefore we may assume that  $V$  is not empty. Note that  $V$  is invariant under the action of  $K_W$ . We denote by  $S$  (like *sail*) the hull of  $W \cup V$  (see Definition 2.14). Let  $W'$  be the  $\sigma/10$ -neighborhood of  $K_S.S$ . In particular  $W'$  contains the  $\sigma/10$ -neighborhood of  $W$ . The goal is to prove that  $W'$  is a windmill.

**Lemma 3.6.** *Let  $v \in v(\mathcal{R})$ . If  $d(v, S) \leq \sigma/5$  then  $v \in S$ .*

*Proof.* By definition of hulls, a  $\delta$ -projection of  $v$  on  $S$  lies on a  $(1, \delta)$ -quasi-geodesics between two points  $y, y'$  of  $W \cup V$ . In particular  $\langle y, y' \rangle_v \leq d(v, S) + 2\delta \leq \sigma/5 + 2\delta$ . We assume that  $y, y' \in V$ . The proof for the other cases works in the same way. Let us denote by  $z$  and  $z'$  respective  $\delta$ -projections of  $y$  and  $y'$  on  $W$ . By hyperbolicity

$$\min \left\{ |y - v| - |y - z|, \langle z, z' \rangle_v, |y' - v| - |y' - z'| \right\} \leq \langle y, y' \rangle_v + 2\delta \leq \frac{\sigma}{5} + 4\delta. \quad (10)$$

Assume first that the minimum is achieved by  $|y - v| - |y - z|$  (the proof works similarly for  $|y' - v| - |y' - z'|$ ). By construction  $|y - z|$  is bounded above by  $3\sigma/10 + \delta$ , thus  $|y - v| < \sigma$ . Nevertheless the distance between two distinct apices of  $\mathcal{R}$  is at least  $\sigma$ . Therefore  $y = v$ . Hence  $v \in V$ . Assume now that the minimum in (10) is achieved by  $\langle z, z' \rangle_v$ . The windmill  $W$  being  $2\delta$ -quasi-convex, we have

$$d(v, W) \leq \langle z, z' \rangle_v + 2\delta \leq \frac{\sigma}{5} + 6\delta \leq \frac{3\sigma}{10}.$$

By definition of  $V$ ,  $v$  is necessarily of point of  $W \cup V$ . □

Recall that  $\mathcal{R}$  is  $G$ -invariant. It follows from Lemma 3.6 that every apex contained in the  $\sigma/5$ -neighborhood of  $K_S.S$  actually belongs to  $S$ . Since  $W'$  is the  $\sigma/10$  neighborhood of  $K_S.S$ , it satisfies (W3). Moreover, all the apices contained in  $W'$  lies in  $K_S.S$ . Hence  $K_{W'} = K_S$ . In particular  $K_S.S$  and thus  $W'$  are  $K_S$ -invariant. This corresponds to (W2).

**Lemma 3.7.** *Let  $(H, v) \in \mathcal{R}$  such that  $v \in V$ . Let  $x, y \in S$  and  $h \in H \setminus \{1\}$ . Then  $\langle x, hy \rangle_v \leq 7\delta$ .*

*Proof.* Recall that  $S$  is the hull of  $W \cup V$ . According to Lemma 2.16, it is sufficient to prove that for all  $x, y \in W \cup V$ ,  $\langle x, hy \rangle_v \leq 4\delta$ . Let  $x, y \in W \cup V$ . Note that if  $x = v$  or  $y = v$ , then  $\langle x, hy \rangle_v = 0$  ( $h$  fixes  $v$ ). Therefore we can assume that  $x$  and  $y$  are distinct from  $v$ . We denote by  $r$  a  $\delta$ -projection of  $v$  on  $W$ . We claim that  $\langle x, v \rangle_r \leq 7\delta$ . If  $x$  belongs to  $W$ , then by Proposition 2.10,  $\langle x, v \rangle_r \leq 3\delta$ . Assume now that  $x$  is a point of  $V \setminus \{v\}$ . Fix  $p$  a  $\delta$ -projection of  $x$  on  $W$ . By Proposition 2.10,

$$|p - r| \leq \max \left\{ |x - v| - |x - p| - |v - r| + 14\delta, 7\delta \right\}.$$

However  $x$  and  $v$  are two distinct apices of  $V$ . It follows that  $|x - v| \geq \sigma$  whereas  $|x - p|$  and  $|v - r|$  are at most  $3\sigma/10 + \delta$ . By triangle inequality  $|p - r| > 7\delta$ . Consequently we necessarily have

$$|x - p| + |p - r| + |r - v| \leq |x - v| + 14\delta$$

In particular  $\langle x, v \rangle_r \leq 7\delta$ , which proves our claim. Similarly  $\langle y, r \rangle_v \leq 7\delta$ . Lemma 3.3 combined with the hyperbolicity condition leads to

$$\min \left\{ \langle r, x \rangle_v, \langle x, hy \rangle_v, \langle hy, hr \rangle_v \right\} \leq \langle r, hr \rangle_v + 2\delta \leq 4\delta. \quad (11)$$

Since  $W$  is a windmill, all the apices of  $\mathcal{R}$  in the  $\sigma/10$ -neighborhood of  $W$  are actually contained in  $W$ . It follows that  $|v - r| > \sigma/10$ . Hence  $\langle x, r \rangle_v = |v - r| - \langle x, v \rangle_r$  is bounded below by  $\sigma/10 - 7\delta$ . The same holds for  $\langle hy, hr \rangle_v = \langle y, r \rangle_v$ . Consequently the minimum in (11) is necessarily achieved by  $\langle x, hy \rangle_v$ . Hence  $\langle x, hy \rangle_v \leq 4\delta$ . □

**Lemma 3.8.** *Let  $y, y' \in S$ . Let  $g \in K_S$ . There exists a sequence of points  $y = y_0, \dots, y_{m+1} = gy'$  of  $X$  satisfying the following properties*



- (i) for all  $i \in \{1, \dots, m+1\}$  there exists  $g_i \in K_S$  such that  $g_i^{-1}y_{i-1}$  and  $g_i^{-1}y_i$  belong to  $S$ ,
- (ii) for all  $i \in \{1, \dots, m-1\}$ ,  $|y_{i+1} - y_i| \geq \sigma$ .
- (iii) for all  $i \in \{1, \dots, m\}$ ,  $\langle y_{i-1}, y_{i+1} \rangle_{y_i} \leq 7\delta$ ,
- (iv) For all  $x \in X$  there exists  $i \in \{0, \dots, m\}$  such that  $\langle y_{i+1}, y_i \rangle_x \leq \langle y, gy' \rangle_x + 22\delta$ .

Moreover, if  $m \leq 1$  then there exists  $(H, v) \in \mathcal{R}$ , such that  $v \in V$  and  $g \in H.K_W$ .

*Proof.* If  $g \in K_W$  then the points  $y_0 = y$  and  $y_1 = gy'$  lie in  $S$  and hence satisfy the conclusion of the lemma. Assume now that  $g$  does not belong to  $K_W$ . The group  $K_S$  is generated by  $K_W$  and the rotation groups of the pairs  $(H, v) \in \mathcal{R}$  where  $v \in V$ . It follows that  $g$  can be written  $g = u_0 h_1 u_1 \dots u_{m-1} h_m u_m$  where  $m \in \mathbb{N}^*$  and

- (i) for all  $i \in \{0, \dots, m\}$ ,  $u_i$  belongs to  $K_W$ ,
- (ii) for all  $i \in \{1, \dots, m\}$ , there exists  $(H_i, v_i) \in \mathcal{R}$  such that  $v_i \in V$  and  $h_i \in H_i \setminus \{1\}$ .

We choose such a decomposition of  $g$  which minimizes  $m$ . We claim that for all  $i \in \{1, \dots, m-1\}$ ,  $u_i v_{i+1} \neq v_i$ . Assume on the contrary that this assertion is false. Using the action of  $G$  on  $\mathcal{R}$ ,  $u_i H_{i+1} u_i^{-1} = H_i$ . Therefore we can write  $h_i u_i h_{i+1} u_{i+1} = (h_i u_i h_{i+1} u_i^{-1})(u_i u_{i+1})$ , where  $h_i u_i h_{i+1} u_i^{-1} \in H_i$  and  $u_i u_{i+1} \in K_W$ . This leads to a shorter decomposition of  $g$ . Contradiction. For all  $i \in \{1, \dots, m\}$ , we put  $g_i = u_0 h_1 u_1 \dots h_{i-1} u_{i-1}$  and  $y_i = g_i v_i$ . Moreover, we put  $g_{m+1} = g$ ,  $y_0 = y$  and  $y_{m+1} = gy'$ . Note that, if  $m = 1$  then  $g$  can be written  $g = u_0 h_1 u_1 = (u_0 h_1 u_0^{-1})(u_0 u_1)$ , where  $u_0, u_1 \in K_W$  and  $h_1 \in H_1$ . However the set  $V$  is invariant under the action of  $K_W$ . Therefore  $u_0 h_1 u_0^{-1}$  is an element of the rotation group  $u_0 H_1 u_0^{-1}$ , whose apex  $u_0 v_1$  belongs to  $V$ . This proves the last assertion of the lemma.

Let  $i \in \{2, \dots, m+1\}$ . By construction  $g_i^{-1}y_i$  is a point of  $S$ . By definition of rotation family  $h_{i-1}$  fixes the apex  $v_{i-1}$ , hence  $g_i^{-1}y_{i-1} = u_{i-1}^{-1} h_{i-1}^{-1} v_{i-1} = u_{i-1}^{-1} v_{i-1}$ . However  $V$  is invariant under the action of  $u_{i-1} \in K_W$ , thus  $g_i^{-1}y_{i-1}$  belongs to  $S$ . On the other hand  $y_0 = y$  and  $y_1 = u_0 v_1$  belong to  $S$ . This completes the proof of Point (i).

Let  $i \in \{1, \dots, m-1\}$ . The apex  $v_i$  is fixed by  $h_i$  therefore

$$|y_{i+1} - y_i| = |g_i h_i u_i v_{i+1} - g_i v_i| = |u_i v_{i+1} - v_i|.$$

However, we explained that  $u_i v_{i+1}$  and  $v_i$  are necessarily two distinct apices of  $\mathcal{R}$ , therefore  $|y_{i+1} - y_i| \geq \sigma$ . This proves Point (ii).

Let  $i \in \{1, \dots, m\}$ . By construction  $g_i^{-1}y_i = v_i$  whereas  $g_i^{-1}y_{i-1}$  belongs to  $S$ . On the other hand  $g_i^{-1}y_{i+1} = g_i^{-1}g_{i+1}g_{i+1}^{-1}y_{i+1} = h_i u_i g_{i+1}^{-1}y_{i+1}$ . However  $u_i$  stabilizes  $S$  which contains  $g_{i+1}^{-1}y_{i+1}$ , hence  $g_i^{-1}y_{i+1}$  belongs to  $h_i S$ . By Lemma 3.7,  $\langle y_{i-1}, y_{i+1} \rangle_{y_i} = \langle g_i^{-1}y_{i-1}, g_i^{-1}y_{i+1} \rangle_{v_i}$  is bounded above by  $7\delta$ , which proves Point (iii). We chose the constants  $\sigma$  and  $\delta$  in such a way that we can apply Proposition 2.7 to the sequence  $y_0, \dots, y_{m+1}$ . Point (iv) follows from the stability of discrete quasi-geodesics.  $\square$

**Lemma 3.9.** *The set  $K_S.S$  is  $28\delta$ -quasi-convex whereas  $W'$  is  $2\delta$ -quasi-convex.*

*Proof.* The set  $W'$  was defined as the  $\sigma/10$ -neighborhood of  $K_S.S$ . According to Lemma 2.11, it is sufficient to show that  $K_S.S$  is  $28\delta$ -quasi-convex. Let  $x \in X$  and  $y, y' \in K_S.S$ . It follows from Lemma 3.8 that there exist  $z, z' \in S$  and  $g \in K_S$  such that  $\langle gz, gz' \rangle_x \leq \langle y, y' \rangle_x + 22\delta$ . However  $S$  being a hull, it is  $6\delta$ -quasi-convex. Therefore

$$d(g^{-1}x, K_S.S) \leq d(g^{-1}x, S) \leq \langle gz, gz' \rangle_x + 6\delta \leq \langle y, y' \rangle_x + 28\delta.$$

By construction  $K_S.S$  is invariant under the action of  $K_S$ . It follows that  $d(x, K_S.S) \leq \langle y, y' \rangle_x + 28\delta$ .  $\square$

**Lemma 3.10.** *Let  $(H, v) \in \mathcal{R}$  such that  $v \in V$ . Let  $h \in H \setminus \{1\}$  and  $u \in K_W \setminus \{1\}$ . For all  $y \in S$ ,  $|huy - y| \geq \sigma - 16\delta$ .*

*Proof.* We need to distinguish two cases.

**Case 1.** *There exists  $(H', v') \in \mathcal{R}$  such that  $v' \in W$  and  $u \in H' \setminus \{1\}$ .* By Lemma 3.3,  $\langle uy, y \rangle_{v'} \leq 2\delta$  i.e.,  $|y - v'| = |uy - v'|$  is bounded above by  $|uy - y|/2 + 2\delta$ . The triangle inequality yields

$$\begin{aligned} |v - v'| &\leq \min \{|v - y| + |y - v'|, |v - uy| + |uy - v'|\} \\ &\leq \min \{|v - y|, |v - uy|\} + \frac{1}{2}|uy - y| + 2\delta \\ &\leq |uy - v| + |v - y| + 2\delta. \end{aligned}$$

However  $v$  and  $v'$  are two distinct apices of  $\mathcal{R}$ , thus  $|v - v'| \geq \sigma$ . It follows that  $|uy - v| + |v - y|$  is at least  $\sigma - 2\delta$ . Recall that  $y$  and  $uy$  are two points of  $S$ . By Lemma 3.7, we obtain

$$|huy - y| \geq |huy - v| + |v - y| - 14\delta = |uy - v| + |v - y| - 14\delta \geq \sigma - 16\delta.$$

**Case 2.** *Assume that  $u$  does not belong to a rotation group.* Let us denote by  $p$  and  $r$  respective  $\delta$ -projections of  $y$  and  $v$  on  $W$ . In particular  $up$  is a  $\delta$ -projection of  $uy$  on  $W$ . By projection on a quasi-convex (Lemma 2.10) we have

$$\begin{aligned} |up - r| &\leq \max \{|uy - v| - |v - r| - |y - p| + 14\delta, 7\delta\} \\ |p - r| &\leq \max \{|y - v| - |v - r| - |y - p| + 14\delta, 7\delta\} \end{aligned}$$

Since  $W$  is a windmill and  $u \in K_W$ , then  $|ur - p| + |p - r| \geq |ur - r| \geq \sigma - 166\delta$ . Therefore the two previous maxima cannot be both achieved by  $7\delta$ . Assume for instance that the first maximum is not achieved by  $7\delta$  (the other case is symmetric). By Lemma 3.3, we get  $|huy - y| \geq |uy - v| + |v - y| - 14\delta$  which leads to

$$|huy - y| \geq |y - p| + |up - r| + |r - v| + |v - y| - 28\delta.$$

However  $r$  is a  $\delta$ -projection of  $v$  on  $W$ , thus

$$|v - y| + |y - p| \geq |v - p| \geq |r - p| - \langle p, v \rangle_r \geq |r - p| - 3\delta.$$

Consequently the previous inequality becomes

$$|huy - y| \geq |up - r| + |r - p| + |r - v| - 31\delta \geq |up - p| + |r - v| - 31\delta.$$

Nevertheless Axiom (W4) for  $W$  gives  $|up - p| \geq \sigma - 166\delta$ . Moreover, by construction  $|v - r| \geq d(x, W) \geq \sigma/10$ . Thus  $|huy - y| \geq \sigma$ .  $\square$

**Lemma 3.11.** *Let  $g \in K_S$  such that for every  $H \in H(\mathcal{R})$ ,  $g$  does not belong to  $H$ . For all  $x \in X$ ,  $|gx - x| \geq \sigma - 166\delta$ .*

*Proof.* Let  $x \in X$ . Since  $W$  is already a windmill we can assume that  $g$  does not belong to  $K_W$ , otherwise the result would follow from (W4). We denote by  $y$  a  $\delta$ -projection of  $x$  on  $K_S.S$ . The rotation family  $\mathcal{R}$  is invariant under the action of  $G$ . Without loss of generality we may assume that  $y \in S$ . We claim that  $|gy - y| \geq \sigma - 48\delta$ . According to Lemma 3.8 there exists a sequence of points  $y = y_0, \dots, y_{m+1} = gy$  satisfying the following conditions

- (i) for all  $i \in \{1, \dots, m-1\}$ ,  $|y_{i+1} - y_i| \geq \sigma$ .
- (ii) for all  $i \in \{1, \dots, m\}$ ,  $\langle y_{i-1}, y_{i+1} \rangle_{y_i} \leq 7\delta$ ,

Assume that  $m \geq 2$ . The constant  $\sigma$  has been chosen to apply the stability of discrete quasi-geodesic. By Proposition 2.7,  $\langle y, gy \rangle_{y_1}$  and  $\langle y_1, gy \rangle_{y_2}$  are at most  $12\delta$ . It follows that

$$|gy - y| \geq |gy - y_2| + |y_2 - y_1| + |y_1 - y| - 108\delta \geq \sigma - 48\delta.$$

Assume now that  $m \leq 1$ . According to Lemma 3.8, there exists  $(H, v) \in \mathcal{R}$  such that  $v \in V$  and  $g \in H.K_W$ . In particular  $g$  can be written  $g = hu$  where  $h \in H$  and  $u \in K_W$ . Since  $g$  does not belong to  $K_W$ ,  $h$  is non-trivial. By assumption  $g$  does not belong to  $H$  thus  $u \neq 1$ . Applying Lemma 3.10,  $|gy - y| \geq \sigma - 16\delta$ , which completes the proof of our claim. By Lemma 3.9,  $K_S.S$  is  $28\delta$ -quasi-convex. Applying Lemma 2.10 we obtain

$$|gx - x| \geq |gx - gy| + |gy - y| + |y - x| - 118\delta \geq \sigma - 166\delta. \quad \square$$

We already proved that Axioms (W2) and (W3) for  $W'$  follow from Lemma 3.6. Axioms (W1) and (W4) respectively correspond to Lemmas 3.9 and 3.11. Hence  $W'$  is a windmill.  $\square$

*Proof of Theorem 3.2.* Let  $g \in K$ . We choose an apex  $v \in v(\mathcal{R})$ . The set  $\{v\}$  is a windmill. Iterating Proposition 3.5, we obtain a windmill  $W$  containing sufficiently many apices of  $v(\mathcal{R})$  so that  $g \in K_W$ . It follows from (W4) that if  $g$  does not belong to any rotation group  $H \in H(\mathcal{R})$  then for every  $x \in X$   $|gx - x| \geq \sigma - 166\delta$ .  $\square$

**Corollary 3.12.** *Let  $l \geq 0$ . Let  $x \in X$  such that for every apex  $v \in v(\mathcal{R})$ ,  $|x - v| \geq l$ . Let  $g \in K \setminus \{1\}$ . Then  $|gx - x| \geq \min\{2l, \sigma/10\}$ .*

**Remark.** In particular the group  $K$  acts freely discontinuously on the space  $X \setminus v(\mathcal{R})$ . By freely discontinuously we mean that for every  $x \in X \setminus v(\mathcal{R})$  there exists  $r > 0$  such that for every  $g \in K$  if  $gB(x, r)$  intersects  $B(x, r)$  then  $g$  is trivial.

*Proof.* If  $g$  does not belong to a rotation group then by Theorem 3.2,  $|gx - x| \geq \sigma - 166\delta$ . Therefore we can assume that there exists  $(H, v) \in \mathcal{R}$  such that  $g \in H \setminus \{1\}$ . If  $|x - v| \geq \sigma/10$ , then by Lemma 3.3,  $|gx - x| \geq 2|x - v| - 4\delta \geq \sigma/10$ . Otherwise the definition of rotation family yields  $|gx - x| = 2|x - v| \geq 2l$ .  $\square$

**Corollary 3.13.** *Let  $(H, v) \in \mathcal{R}$ . We have  $\text{Stab}(v) \cap K = H$ . Moreover, for every  $x \in B(v, \sigma/5)$ , for every  $g \in K \setminus H$  we have  $|gx - x| > 3\sigma/5$ .*

*Proof.* By construction  $H$  lies in  $\text{Stab}(v) \cap K$ . Let  $g \in \text{Stab}(v) \cap K$  which is not trivial. In particular  $gv = v$ . By Theorem 3.2, there exists  $(H', v') \in \mathcal{R}$  such that  $g \in H' \setminus \{1\}$ . Assume now that  $v \neq v'$ . By Lemma 3.3,  $|gv - v| \geq 2|v - v'| - 4\delta > 0$ . Contradiction. Hence  $v = v'$  and  $g$  belongs to  $H$ .

Let us consider now  $x \in B(v, \sigma/5)$  and  $g \in K \setminus H$ . Assume that our second assertion is false. Then by triangle inequality  $|gv - v| \leq |gx - x| + 2|x - v| < \sigma$ . However the distance between two distinct apices is at least  $\sigma$ . Therefore  $g$  belongs to  $\text{Stab}(v) \cap K = H$ . Contradiction.  $\square$

### 3.2 Consequences

We keep here the notations and assumptions of the previous section. In particular,  $G$  is a group acting by isometries on the  $\delta$ -hyperbolic length space  $X$  and  $\mathcal{R}$  is a  $\sigma$ -rotation family with  $\sigma \geq \sigma_0$  where  $\sigma_0$  is the parameter given by Theorem 3.2. The space  $\bar{X}$  is defined to be the quotient of  $X$  by  $K$ . If  $x$  is a point of  $X$  we denote by  $\bar{x}$  its image by the canonical map  $\nu : X \rightarrow \bar{X}$ . Moreover, we write  $\bar{v}(\mathcal{R})$  for the image of  $v(\mathcal{R})$  in  $\bar{X}$ . We endow  $\bar{X}$  with the pseudo-metric defined in the following way.

$$\forall x, x' \in X, \quad |\bar{x} - \bar{x}'|_{\bar{X}} = \inf_{g \in K} |gx - x'|_X.$$

By construction the quotient  $\bar{G} = G/K$  acts by isometries on  $\bar{X}$ . Given an element  $g$  of  $G$  we denote by  $\bar{g}$  its image by the projection  $\pi : G \rightarrow \bar{G}$ . It follows from Corollary 3.13 that for every  $(H, v) \in \mathcal{R}$  this map induces an embedding  $\text{Stab}(v)/H \hookrightarrow \bar{G}$ . Furthermore  $X \setminus v(\mathcal{R})$  is a covering space of  $\bar{X} \setminus \bar{v}(\mathcal{R})$ .

**Proposition 3.14.**  *$\bar{X}$  is a metric length space.*

*Proof.* One only needs to prove that the pseudo-distance on  $\bar{X}$  is definite positive. The fact that  $\bar{X}$  is a length space follows from the length structure on  $X$  [4, Chap. I.5, Lemma 5.20]. Let  $x$  and  $x'$  be two points of  $X$  such that  $|\bar{x} - \bar{x}'| = 0$ . Assume first that  $x$  is not an apex. The set of apices  $v(\mathcal{R})$  is closed and  $G$ -invariant, thus neither is  $x'$ . Since  $K$  acts freely discontinuously on  $X \setminus v(\mathcal{R})$ ,  $x$  and  $x'$  are necessarily in the same  $K$ -orbit. Assume now that  $x$  and  $x'$  are both apices. By definition of the pseudo-metric there exists  $g \in K$  such that  $|gx' - x| < \sigma$ . However the distance between two distinct apices is at least  $\sigma$ . Hence  $gx = x'$ . Consequently, in both cases  $\bar{x} = \bar{x}'$ .  $\square$

**Proposition 3.15.** *Let  $r \in (0, \sigma/40]$ . Let  $x \in X$  such that for all  $v \in v(\mathcal{R})$ ,  $|x - v| \geq 2r$ . The map  $\nu : X \rightarrow \bar{X}$  induces an isometry from  $B(x, r)$  onto  $B(\bar{x}, r)$ .*

**Remark.** In particular, the map  $\nu : X \rightarrow \bar{X}$  induces a local isometry from  $X \setminus v(\mathcal{R})$  onto  $\bar{X} \setminus \bar{v}(\mathcal{R})$ .

*Proof.* By construction the ball  $B(x, 2r)$  is contained in  $X \setminus v(\mathcal{R})$ . According to Corollary 3.12,  $\nu$  induces a bijection from  $B(x, 2r)$  onto its image. It follows that it induces an isometry from  $B(x, r)$  onto its image which is exactly  $B(\bar{x}, r)$ .  $\square$

**Lemma 3.16.** *For every  $v \in v(\mathcal{R})$  the ball  $B(\bar{v}, \sigma/5)$  of  $\bar{X}$  is  $2\delta$ -hyperbolic.*

*Proof.* First note that for every point  $x \in B(v, \sigma/5)$  we have  $|\bar{x} - \bar{v}| = |x - v|$  (this is a consequence of Corollary 3.13). Let  $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$  be three points of  $B(\bar{v}, \sigma/5)$ . We denote by  $x$ ,  $y$  and  $z$  respective pre-images of  $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$  in  $B(v, \sigma/5)$ . By definition of the metric there exists two sequences  $(g_n)$  and  $(h_n)$  of elements of  $K$  such that  $|g_n x - y|$  and  $|h_n z - y|$  respectively converge to  $|\bar{x} - \bar{y}|$  and  $|\bar{z} - \bar{y}|$  as  $n$  approaches infinity. In particular if  $n$  is sufficiently large we get by triangle inequality  $|g_n v - v| < \sigma$ . It follows that  $g_n$  belongs to the stabilizer of  $v$  and thus  $g_n x$  also lies in  $B(v, \sigma/5)$ . In

the same way for  $n$  sufficiently large  $h_n z$  belongs to  $B(v, \sigma/5)$ . However  $X$  is  $\delta$ -hyperbolic, therefore  $g_n x, y, h_n z$  and  $v$  satisfy the four points inequality (1). Consequently

$$\langle \bar{x}, \bar{z} \rangle_{\bar{v}} \geq \langle g_n x, h_n z \rangle_v \geq \min \{ \langle g_n x, y \rangle_v, \langle y, h_n z \rangle_v \} - \delta.$$

Taking the limit as  $n$  approaches infinity we obtain

$$\langle \bar{x}, \bar{z} \rangle_{\bar{v}} \geq \min \{ \langle \bar{x}, \bar{y} \rangle_{\bar{v}}, \langle \bar{y}, \bar{z} \rangle_{\bar{v}} \} - \delta.$$

By Lemma 2.1,  $B(\bar{x}, \sigma/5)$  is  $2\delta$ -hyperbolic. □

**Lemma 3.17.** *The space  $\bar{X}$  is  $50\delta$ -simply connected i.e., its fundamental group is normally generated by free homotopies of loops of diameter at most  $50\delta$ .*

*Proof.* Let  $\bar{\gamma}$  be a loop in  $\bar{X}$  based at  $\bar{x}$  and  $\gamma_1$  a lift of  $\bar{\gamma}$  in  $X$ . If  $x$  is the initial point of  $\gamma$  there exists  $g \in K$  such that the terminal point of  $\gamma_1$  is  $gx$ . Note that  $K$  is generated by elliptic isometries of  $X$ . Moreover, the translation length of an elliptic isometry of  $X$  is at most  $32\delta$ . Therefore there exists an other path  $\gamma_2$  of  $X$  joining  $x$  to  $gx$  such that its image in  $\bar{X}$  can be written as a product of loops of diameter at most  $32\delta$ . The space  $X$  being  $\delta$ -hyperbolic its Rips complex  $P_{4\delta}(X)$  is simply-connected [6, Chap. 5, Prop. 1.1]. It implies that  $X$  is  $50\delta$ -simply-connected. On the other hand  $\gamma_2^{-1}\gamma_1$  is a loop of  $X$ . Consequently it can be written in  $X$  and thus in  $\bar{X}$  as a product of loops of diameter at most  $50\delta$ . Hence  $\bar{X}$  is  $50\delta$ -simply-connected. □

**Proposition 3.18.**  *$\bar{X}$  is  $600\delta$ -hyperbolic*

*Proof.* According to Propositions 3.15 and 3.16 every ball  $\bar{B}$  of radius  $\sigma/40$  in  $\bar{X}$  is  $2\delta$ -hyperbolic. Indeed if the distance between center of  $\bar{B}$  and every apex is at least  $\sigma/10$  then it is isometric to a ball of  $X$  which is  $\delta$ -hyperbolic. Otherwise there is  $v \in v(\mathcal{R})$  such that  $\bar{B}$  lies in  $B(\bar{v}, \sigma/5)$  which is  $2\delta$ -hyperbolic. On the other hand, by Proposition 3.17,  $\bar{X}$  is  $50\delta$ -simply-connected. Recall that we chose  $\sigma \geq \sigma_0$  with  $\sigma_0 > 10^{20}\delta$ . Therefore we can apply the Cartan-Hadamard Theorem (see Theorem A.1). It follows that  $\bar{X}$  is  $600\delta$ -hyperbolic. □

We now explain how to lift figures of  $\bar{X}$  in  $X$ . To that end, we first introduce a very general construction, coming from the theory of covering spaces, to lift a path. Let  $\bar{x}$  be a point of  $\bar{X}$  and  $x$  a preimage of  $\bar{x}$  in  $X$ . Let  $\bar{\gamma} : I \rightarrow \bar{X}$  be a path starting at  $\bar{x}$ . We assume that  $\bar{\gamma}$  does not go through  $\bar{v}(\mathcal{R})$ . Since  $K$  acts freely discontinuously on  $X \setminus v(\mathcal{R})$ , there exists a unique path  $\gamma : I \rightarrow X$  starting at  $x$  and lifting  $\bar{\gamma}$ . We write  $x + \bar{\gamma}$  for the terminal point of  $\gamma$ . It is a preimage of the terminal point of  $\bar{\gamma}$  that we denote  $\bar{x} + \bar{\gamma}$ . If  $\bar{\gamma}_1$  and  $\bar{\gamma}_2$  are two paths with the same extremities and homotopic relative to their endpoints in  $\bar{X} \setminus \bar{v}(\mathcal{R})$  then  $x + \bar{\gamma}_1$  and  $x + \bar{\gamma}_2$  define the same point of  $X$ .

In [10] the authors make an intensive use of this topological point of view. They need nevertheless to work with orbifolds to deal with the torsion. For our purpose we prefer a more geometrical approach that follows from the rotation families. The statements in the remainder of this section corresponds to Lemme 5.9.4 and Lemme 5.10.1 of [10].

Since the map  $X \setminus v(\mathcal{R}) \rightarrow \bar{X} \setminus \bar{v}(\mathcal{R})$  is a local isometry,  $\bar{\gamma}$  and its lift  $\gamma$  have the same length. If  $\bar{\gamma}$  is a  $(1, l)$ -quasi-geodesic, then

$$L(\gamma) = L(\bar{\gamma}) \leq |(\bar{x} + \bar{\gamma}) - \bar{x}| + l \leq |(x + \bar{\gamma}) - x| + l.$$

Here  $L(\gamma)$  and  $L(\bar{\gamma})$  stand for the respective lengths of the paths  $\gamma$  and  $\bar{\gamma}$ . In particular  $\gamma$  is also a  $(1, l)$ -quasi-geodesic. Moreover, we have  $|(x + \bar{\gamma}) - x| \leq |(\bar{x} + \bar{\gamma}) - \bar{x}| + l$ .

**Lemma 3.19.** *Let  $\bar{x}$  be a point of  $\bar{X}$  and  $x$  a preimage of  $\bar{x}$  in  $X$ . Let  $\bar{\gamma}_1$ ,  $\bar{\gamma}_2$  and  $\bar{\gamma}_3$  be  $(1, \delta)$ -quasi-geodesic paths satisfying the following:*

- (i)  $\bar{\gamma}_1$  and  $\bar{\gamma}_3$  start at  $\bar{x}$ ,
- (ii)  $\bar{\gamma}_2$  starts at  $\bar{x} + \bar{\gamma}_1$  and ends at  $\bar{x} + \bar{\gamma}_3$ ; see Figure 1(a).

Assume that for every  $v \in v(\mathcal{R})$ , these paths do not enter the ball  $B(\bar{v}, \sigma/10 + 7\delta)$ . Then  $(x + \bar{\gamma}_1) + \bar{\gamma}_2 = x + \bar{\gamma}_3$ .

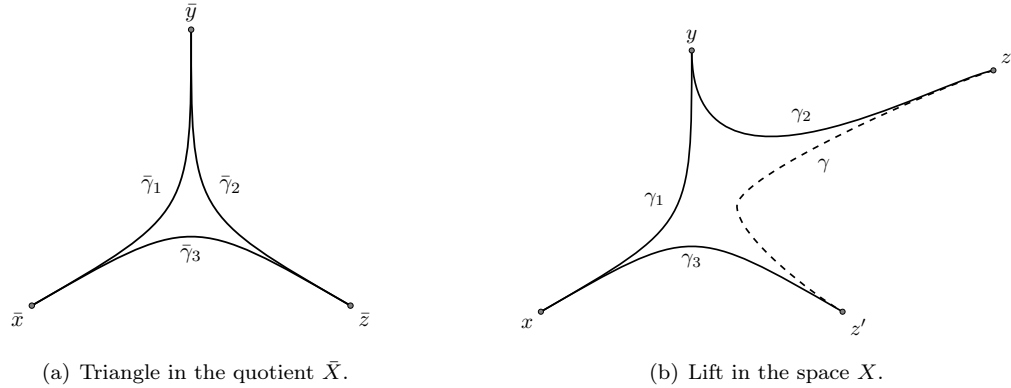


Figure 1: Lifting a triangle

*Proof.* For simplicity of notation, let us respectively denote by  $y$ ,  $z$  and  $z'$  the points  $x + \bar{\gamma}_1$ ,  $y + \bar{\gamma}_2$  and  $x + \bar{\gamma}_3$ . See Figure 1(b). In particular  $z$  and  $z'$  have the same image in  $\bar{X}$ . We write  $\gamma_1$  (respectively  $\gamma_2$ ,  $\gamma_3$ ) for the lift of  $\bar{\gamma}_1$  (respectively  $\bar{\gamma}_2$ ,  $\bar{\gamma}_3$ ) which starts at  $x$  (respectively  $y$ ,  $x$ ). These paths are  $(1, \delta)$ -quasi-geodesics. Moreover, for all  $v \in v(\mathcal{R})$  they do not enter  $B(v, \sigma/10 + 7\delta)$ . Let  $\eta \in (0, \delta)$ . We denote by  $\gamma$  a  $(1, \eta)$ -quasi-geodesic of  $X$  joining  $z$  to  $z'$ . By hyperbolicity,  $\gamma$  is contained in the  $(\eta + 6\delta)$ -neighborhood of  $\gamma_1 \cup \gamma_2 \cup \gamma_3$ . Consequently, if  $\eta$  is sufficiently small,  $\gamma$  does not intersect any of the balls  $B(v, \sigma/10)$  where  $v \in v(\mathcal{R})$ . According to Proposition 3.15 the path  $\bar{\gamma}$ , image of  $\gamma$  in  $\bar{X}$ , is a  $\sigma/25$ -local  $(1, \eta)$ -quasi-geodesic whose length is the same as the one of  $\gamma$ . However we chose  $\sigma$  in such a way that we can apply the stability of quasi-geodesics. In particular  $\bar{\gamma}$  is a (global)  $(2, \eta)$ -quasi-geodesic. Its endpoints are the same, namely  $\bar{z} = \bar{z}'$ . Therefore its length is at most  $\eta$ . Consequently  $|z - z'| \leq L(\gamma) \leq \eta$ . This inequality holds for every  $\eta > 0$ , hence  $z = z'$  i.e.,  $(x + \bar{\gamma}_1) + \bar{\gamma}_2 = x + \bar{\gamma}_3$ .  $\square$

**Proposition 3.20.** *Let  $\alpha \geq 0$ . Let  $\bar{Y}$  be a  $\alpha$ -quasi-convex subset of  $\bar{X}$  such that for every  $v \in v(\mathcal{R})$ ,  $\bar{Y}$  does not intersect  $B(\bar{v}, \sigma/10 + \alpha + 8\delta)$ . Let  $\bar{y}_0$  be a point of  $\bar{Y}$  and  $y_0$  a preimage of  $\bar{y}_0$  in  $X$ . There exists a subset  $Y$  of  $X$  containing  $y_0$  such that the map  $\nu : X \rightarrow \bar{X}$  induces an isometry from  $Y$  onto  $\bar{Y}$ .*

*Proof.* We construct  $Y$  in the following way. Let  $\bar{y}$  be a point of  $\bar{Y}$ . We choose a  $(1, \delta)$ -quasi-geodesic  $\bar{\gamma}$  joining  $\bar{y}_0$  to  $\bar{y}$ . Since  $\bar{Y}$  is  $\alpha$ -quasi-convex,  $\bar{\gamma}$  lies in the  $(\alpha + \delta)$ -neighborhood of  $\bar{Y}$ . In particular, it does not go through a vertex of  $\bar{X}$ . More precisely  $\bar{\gamma}$  does not run through any ball  $B(\bar{v}, \sigma/10 + 7\delta)$  where  $v \in v(\mathcal{R})$ . Consequently,  $y_0 + \bar{\gamma}$  is well-defined. We define  $Y$  to be the set of all points  $y_0 + \bar{\gamma}$  obtained in this way. By construction  $\nu$  maps  $Y$  onto  $\bar{Y}$ . Hence it is sufficient to prove that the restriction of  $\nu$  to  $Y$  preserves the distances. Let  $y_1$  and  $y_2$  be two points of  $Y$ . By definition there exist  $(1, \delta)$ -quasi-geodesics  $\bar{\gamma}_1$  and  $\bar{\gamma}_2$  respectively joining  $\bar{y}_0$  to  $\bar{y}_1$  and  $\bar{y}_2$  such that  $y_1 = y_0 + \bar{\gamma}_1$  and  $y_2 = y_0 + \bar{\gamma}_2$ . Let  $\eta \in (0, \delta)$  and  $\bar{\gamma}$  be a  $(1, \eta)$ -quasi-geodesic joining  $\bar{y}_1$  to  $\bar{y}_2$ . According to Lemma 3.19,  $(y_0 + \bar{\gamma}_1) + \bar{\gamma} = y_0 + \bar{\gamma}_2$  i.e.,  $y_1 + \bar{\gamma} = y_2$ . Hence for every sufficiently small  $\eta > 0$ ,  $|y_1 - y_2| \leq |\bar{y}_1 - \bar{y}_2| + \eta$ . It follows that  $|y_1 - y_2| = |\bar{y}_1 - \bar{y}_2|$ .  $\square$

**Proposition 3.21.** *Let  $\alpha \geq 0$  and  $d \geq \alpha$ . Let  $\bar{Y}$  be an  $\alpha$ -quasi-convex subset of  $\bar{X}$  such that for every  $v \in v(\mathcal{R})$ ,  $\bar{Y}$  does not intersect  $B(\bar{v}, \sigma/10 + d + 1208\delta)$ . Let  $\bar{y}_0$  be a point of  $\bar{Y}$  and  $y_0$  a preimage of  $\bar{y}_0$  in  $X$ . There exists a subset  $Y$  of  $X$  containing  $y_0$  with the following properties:*

- (i) *the map  $\nu : X \rightarrow \bar{X}$  induces an isometry from  $Y$  onto  $\bar{Y}$ ,*
- (ii) *for every  $\bar{g} \in \bar{G}$  such that  $\bar{g}\bar{Y}$  lies in the  $d$ -neighborhood of  $\bar{Y}$  there exists a preimage  $g \in G$  of  $\bar{g}$  with the following property. For every  $y, z \in Y$ ,  $|gy - z| = |\bar{g}\bar{y} - \bar{z}|$ .*

*In particular the projection  $\pi : G \rightarrow \bar{G}$  induces an isomorphism from  $\text{Stab}(Y)$  onto  $\text{Stab}(\bar{Y})$ .*

*Proof.* We denote by  $\bar{Z}$  the  $d$ -neighborhood of  $\bar{Y}$ . Recall that  $\bar{X}$  is  $600\delta$ -hyperbolic. By Proposition 2.11,  $\bar{Z}$  is  $1200\delta$ -quasi-convex, thus it satisfies the assumptions of Proposition 3.20. As in this proposition we construct a subset  $Z$  of  $X$  containing  $y_0$  such that the map  $\nu : X \rightarrow \bar{X}$  induces an isometry from  $Z$  onto  $\bar{Z}$ . We write  $Y$  for the preimage of  $\bar{Y}$  in  $Z$ . In particular  $\nu$  maps  $Y$  isometrically onto  $\bar{Y}$ . Let  $\bar{g} \in \bar{G}$  such that  $\bar{g}\bar{Y} \subset \bar{Z}$ . By construction, there exists  $g \in G$  such that  $gy_0$  is the unique preimage of  $\bar{g}\bar{y}_0$  in  $Z$ . Let  $y \in Y$ . By assumption  $\bar{g}\bar{y}$  is point of  $\bar{Z}$ . We claim that  $gy$  is the (unique) preimage of  $\bar{g}\bar{y}$  in  $Z$ . For simplicity of notation we put  $\bar{y}_1 = \bar{g}\bar{y}_0$  and  $\bar{y}_2 = \bar{g}\bar{y}$ . We denote by  $y_1 = gy_0$  and  $y_2$  their respective preimages in  $Z$ . There exists a  $(1, \delta)$ -quasi-geodesic  $\bar{\gamma}$  (respectively  $\bar{\gamma}_1, \bar{\gamma}_2$ ) joining  $\bar{y}_0$  to  $\bar{y}$  (respectively  $\bar{y}_1, \bar{y}_2$ ) such that  $y = y_0 + \bar{\gamma}$ , (respectively  $y_1 = y_0 + \bar{\gamma}_1$ ,  $y_2 = y_0 + \bar{\gamma}_2$ ). Note that  $\bar{g}\bar{\gamma}$  is a  $(1, \delta)$ -quasi-geodesic joining  $\bar{y}_1 = \bar{g}\bar{y}_0$  to  $\bar{y}_2 = \bar{g}\bar{y}$ . Hence by Lemma 3.19,  $(y_0 + \bar{\gamma}_1) + \bar{g}\bar{\gamma} = y_0 + \bar{\gamma}_2$  i.e.,  $gy_0 + \bar{g}\bar{\gamma} = y_2$ . However  $g\bar{\gamma}$  is exactly the lift of  $\bar{g}\bar{\gamma}$  starting at  $gy_0$  thus  $y_2 = gy$  which proves our claim. Point (ii) follows from the fact that  $\nu$  induces an isometry from  $Z$  onto  $\bar{Z}$ .

Point (ii) implies that the projection  $\pi : G \rightarrow \bar{G}$  maps  $\text{Stab}(Y)$  onto  $\text{Stab}(\bar{Y})$ . We prove now that this map is also one-to-one. Let  $g \in \text{Stab}(Y)$  whose image in  $\text{Stab}(\bar{Y})$  is trivial i.e.,  $g \in K$ . The point  $gy_0$  belongs to  $Y$ . Moreover, this is a lift of  $\bar{g}\bar{y}_0 = \bar{y}_0$ . Using the isometry between  $Y$  and  $\bar{Y}$  we get that  $gy_0 = y_0$ . Since  $K$  acts freely on  $X \setminus v(\mathcal{R})$  and  $y_0$  is not a vertex of  $X$  we get  $g = 1$ .  $\square$

## 4 Cone over a metric space

### 4.1 Definition and metric.

In this section we fix a number  $\rho > 0$ . Its value will be made precise later. It should be thought of as a very large parameter.

**Definition 4.1.** Let  $Y$  be a metric space. The *cone of radius  $\rho$  over  $Y$* , denoted by  $Z(Y)$  is the topological quotient of  $Y \times [0, \rho]$  by the equivalence relation that identifies all the points of the form  $(y, 0)$ .

**Notations.** The equivalence class of  $(y, 0)$ , denoted by  $v$  is called the *apex* of the cone. By abuse of notation, we still write  $(y, r)$  for the equivalence class of  $(y, r)$ .

**Proposition 4.2** (see [4, Chap. I.5, Prop. 5.9]). *The cone  $Z(Y)$  is endowed with a metric characterized in the following way. Let  $x = (y, r)$  and  $x' = (y', r')$  be two points of  $Z(Y)$  then*

$$\operatorname{ch} |x - x'| = \operatorname{ch} r \operatorname{ch} r' - \operatorname{sh} r \operatorname{sh} r' \cos \theta(y, y'),$$

where  $\theta(y, y')$  is the angle at the apex defined by  $\theta(y, y') = \min \{\pi, |y - y'| / \operatorname{sh} \rho\}$ . Moreover, if  $Y$  is a length space, so is  $Z(Y)$ .

In fact, the cone  $Z(Y)$  is the ball of radius  $\rho$  of the cone  $C_{-1}(Y/\pi \operatorname{sh} \rho)$  defined in [4, Chap. I.5]. The distance between two points  $x = (y, r)$  and  $x' = (y', r')$  of  $Z(Y)$  has the following geometric interpretation. Consider a geodesic triangle in the hyperbolic plane  $\mathbf{H}_2$  such that lengths of two sides are respectively  $r$  and  $r'$  and the angle between them is  $\theta(y, y')$ . According to the law of cosines,  $|x - x'|$  is exactly the length of the third side of the triangle (see Figure 2).

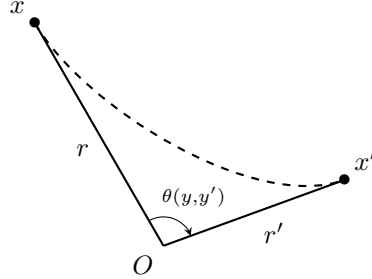


Figure 2: Geometric interpretation of the distance in the cone.

**Proposition 4.3** (see [4, Chap. I.5, Prop. 5.10]). *Let  $x = (y, r)$  and  $x' = (y', r')$  be two points of  $Z(Y)$ .*

- *If  $r, r' > 0$  and  $|y - y'| < \pi \operatorname{sh} \rho$  then there is a one-to-one correspondence between the geodesics of  $Y$  joining  $y$  and  $y'$  and the geodesics of  $Z(Y)$  between  $x$  and  $x'$ .*
- *In all other cases  $|x - x'| = r + r'$ . Moreover, there is a unique geodesic between  $x$  and  $x'$ . It goes through the apex of the cone.*



**Examples.** If  $Y$  is a circle of perimeter  $2\pi \operatorname{sh} \rho$  endowed with the length metric then  $Z(Y)$  is the hyperbolic disc of  $\mathbf{H}_2$  of radius  $\rho$ . If  $Y$  is the real line, then  $Z(Y) \setminus \{v\}$  is the universal cover of the punctured hyperbolic disc of radius  $\rho$ .

In order to compare the space  $Y$  and its cone we introduce the map  $\iota : Y \rightarrow Z(Y)$  which sends  $y$  to  $(y, \rho)$ . It follows from the definition of the metric on  $Z(Y)$  that for all  $y, y' \in Y$ ,  $|\iota(y) - \iota(y')|_{Z(Y)} = \mu(|y - y'|_Y)$ , where  $\mu$  is a map from  $\mathbf{R}_+$  into  $\mathbf{R}_+$  characterized by

$$\forall t \geq 0, \quad \operatorname{ch} \mu(t) = \operatorname{ch}^2 \rho - \operatorname{sh}^2 \rho \cos \left( \min \left\{ \pi, \frac{t}{\operatorname{sh} \rho} \right\} \right).$$

**Proposition 4.4.** *The map  $\mu$  is continuous, concave, non-decreasing. Moreover, we have the followings.*

(i) for all  $t \geq 0$ ,  $t - \frac{1}{24} \left( 1 + \frac{1}{\operatorname{sh}^2 \rho} \right) t^3 \leq \mu(t) \leq t$ .

(ii) for all  $t \in [0, \pi \operatorname{sh} \rho]$ ,  $t \leq \pi \operatorname{sh}(\mu(t)/2)$ .

*Proof.* The shape of the graph of  $\mu$  is given on Figure 3. The proof is left to the reader.  $\square$

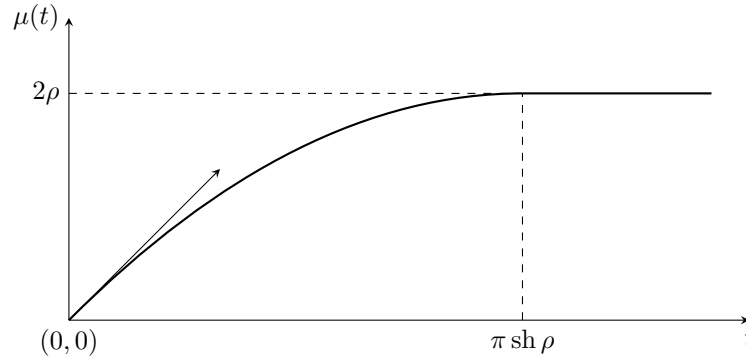


Figure 3: Graph of  $\mu$ .

## 4.2 Hyperbolicity of a cone.

**Proposition 4.5** (Berestovskii see [4, Chap. II.3, Th. 3.14]). *Let  $Y$  be a metric space. If  $Y/\pi \operatorname{sh} \rho$  is  $\operatorname{CAT}(1)$  then the cone  $Z(Y)$  is  $\operatorname{CAT}(-1)$ . In particular if  $Y$  is a tree then  $Z(Y)$  is  $\delta$ -hyperbolic.*

**Remark.** Recall that  $\delta$  is the hyperbolicity constant of the hyperbolic plane  $\mathbf{H}_2$ .

**Proposition 4.6.** *Let  $Y$  be a metric space. The cone  $Z(Y)$  is  $2\delta$ -hyperbolic.*

*Proof.* Every triple of points of  $Y$  can be isometrically embedded into a tripod. Therefore by Proposition 4.5 the cone over three points of  $Y$  is isometrically embedded into a  $\delta$ -hyperbolic space. Consequently, for every triple of points in  $Z(Y)$ , these points together with the apex  $v$  of the cone satisfy the four points inequality (1). The proposition follows then from Lemma 2.1.  $\square$

### 4.3 Group action on a cone

Let  $Y$  be a metric space endowed with an action by isometries of a group  $H$ . This action naturally extends to an action by isometries on  $Z(Y)$  in the following way. For every point  $x = (y, r)$  of  $Z(Y)$ , for every  $h \in H$  we put  $hx = (hy, r)$ . Note that  $H$  fixes the apex  $v$  of the cone. Therefore this action is not necessarily proper (even if the action of  $H$  on  $Y$  is). One should think that  $H$  acts on  $Z(Y)$  as a rotation group with apex  $v$ .

**Lemma 4.7.** *Assume that for every  $h \in H$ ,  $[h] \geq \pi \operatorname{sh} \rho$ . Then for every point  $x \in Z(Y)$ , for every  $h \in H \setminus \{1\}$ ,  $|hx - x| = 2|x - v|$ .*

*Proof.* We denote by  $(y, r)$  the point  $x$ . By assumption  $|hy - y| \geq \pi \operatorname{sh} \rho$ , hence  $\theta(y, y') = \pi$ . Therefore  $|hx - x| = 2r = 2|x - v|$ .  $\square$

We now assume that the action of  $H$  on  $Y$  is proper. We denote by  $\bar{Y}$  the quotient  $Y/H$ . For all  $y \in Y$ , we write  $\bar{y}$  for the image of  $y$  in  $\bar{Y}$ . The space  $\bar{Y}$  is endowed with a metric defined by  $|\bar{y} - \bar{y}'| = \inf_{h \in H} |y - hy'|$ . As we previously explained, the action of  $H$  on  $Z(Y)$  may not be proper. Nevertheless the formula  $|\bar{x} - \bar{x}'| = \inf_{h \in H} |x - hx'|$  still defines a metric on  $Z(Y)/H$ . Moreover, the spaces  $Z(Y)/H$  and  $Z(Y)/H$  are isometric.

**Lemma 4.8.** *Let  $l \geq 2\pi \operatorname{sh} \rho$ . We assume that for every  $h \in H \setminus \{1\}$ ,  $[h] \geq l$ . Let  $x = (y, r)$  and  $x' = (y', r')$  be two points of  $Z(Y)$ . If  $|y - y'|_Y \leq l - \pi \operatorname{sh} \rho$  then  $|\bar{x} - \bar{x}'| = |x - x'|$ .*

*Proof.* Since  $Z(Y)/H$  and  $Z(Y)/H$  are isometric, the distance between  $\bar{x}$  and  $\bar{x}'$  in  $Z(Y)/H$  is given by

$$\operatorname{ch}(|\bar{x} - \bar{x}'|) = \operatorname{ch} r \operatorname{ch} r' - \operatorname{sh} r \operatorname{sh} r' \cos \left( \min \left\{ \pi, \frac{|\bar{y} - \bar{y}'|}{\operatorname{sh} \rho} \right\} \right).$$

If  $|y - y'| < l/2$ , then we have  $|\bar{y} - \bar{y}'| = |y - y'|$ . It follows that  $|\bar{x} - \bar{x}'| = |x - x'|$ . Assume now that  $|y - y'| \geq l/2$ . In particular  $|y - y'| \geq \pi \operatorname{sh} \rho$ . Thus  $|x - x'| = r + r'$ . On the other hand, using the triangle inequality, for all  $h \in H \setminus \{1\}$ ,  $|y - hy'| \geq l - |y - y'|$ , thus  $|\bar{y} - \bar{y}'| \geq \pi \operatorname{sh} \rho$ . Consequently,  $|\bar{x} - \bar{x}'| = r + r' = |x - x'|$ .  $\square$

## 5 Cone-off construction

The goal of this section is to construct a metric space called *cone-off* obtained by attaching a family  $\mathcal{Z}$  of cones on a base space  $X$ . In particular we would like to understand its curvature. Nevertheless during the exposition we will never use the fact that the spaces we attach are cones. Therefore we explain the ideas in a more general situation. In this process the spaces  $Z \in \mathcal{Z}$  are not attached according to an isometry. Therefore one needs a way to measure the distortion between the glued spaces and the base  $X$ . This is the role of the comparison map defined below.

**Definition 5.1.** Let  $a > 0$ . An  $a$ -comparison map is a non-decreasing, concave map  $\mu : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  such that for all  $t \in \mathbf{R}_+$ ,  $t - at^3 \leq \mu(t) \leq t$ .

It follows immediately that for all  $t \geq 0$ ,  $\mu(t) = 0$  if and only if  $t = 0$ . Moreover,  $\mu$  is subadditive, i.e. for all  $s, t \in \mathbf{R}_+$ ,  $\mu(s+t) \leq \mu(s) + \mu(t)$ . Hence  $\mu$  is 1-Lipschitz. The map  $\mu$  studied in Proposition 4.4 is an  $a$ -comparison map with  $a = (1 + 1/\operatorname{sh}^2 \rho)/24$ .

**Definition 5.2.** Let  $a > 0$  and  $\mu$  be an  $a$ -comparison map. Let  $X$  be a metric space. A  $(\mu, X)$ -family  $\mathcal{Z}$  is a collection of triples  $(Z, Y, \iota)$  where  $Z$  is a metric length space and  $\iota$  a map from a non-empty subset  $Y$  of  $X$  into  $Z$  such that  $\iota(Y)$  is closed in  $Z$  and for all  $y, y' \in Y$ ,

$$\mu(|y - y'|_X) \leq |\iota(y) - \iota(y')|_Z. \quad (12)$$

**Definition 5.3.** Let  $a > 0$  and  $\mu$  be an  $a$ -comparison map. Let  $X$  be a metric space and  $\mathcal{Z}$  a  $(\mu, X)$ -family. The *cone-off over  $X$  relatively to  $\mathcal{Z}$*  denoted by  $\dot{X}(\mathcal{Z})$  (or simply  $\dot{X}$ ) is obtained by attaching for all  $(Z, Y, \iota) \in \mathcal{Z}$  the space  $Z$  on  $X$  along  $Y$  according to  $\iota$ .

In other words the space  $\dot{X}$  is the quotient of the disjoint union of  $X$  and all the  $Z$ 's by the equivalence relation which identifies every point  $y \in Y$  with its image  $\iota(y) \in Z$ . To simplify the notations, we use the same letter to design a point of this disjoint union and its equivalence class in  $\dot{X}$ . For the moment  $\dot{X}$  is just a set of points. Our goal is to define a metric on  $\dot{X}$  and study its properties.

### 5.1 Metric on the cone-off

We endow the disjoint union of  $X$  and all the  $Z$ 's with the distance induced by  $|\cdot|_X$  and  $|\cdot|_Z$ . This metric is not necessarily finite: the distance between two points in distinct components is infinite. Let  $x$  and  $x'$  be two points of  $\dot{X}$ . We define  $\|x - x'\|$  to be the infimum over the distances between two points in the previous disjoint union whose classes in  $\dot{X}$  are respectively  $x$  and  $x'$ .

#### Remarks.

- (i) Let  $(Z, Y, \iota) \in \mathcal{Z}$ . If  $x \in Z \setminus Y$  and  $x' \notin Z$ , then  $\|x - x'\| = +\infty$ . In particular  $\|\cdot\|$  is not a distance on  $\dot{X}$  (it does not satisfy the triangle inequality).
- (ii) Let  $x$  and  $x'$  be two points of  $X$ . Using the properties of  $\mu$  combined with (12) we get

$$\mu(|x - x'|_X) \leq \|x - x'\| \leq |x - x'|_X.$$

**Definition 5.4** (Chain between two points). Let  $x$  and  $x'$  be two points of  $\dot{X}$ . A *chain* between  $x$  and  $x'$  is a finite sequence  $C = (z_1, \dots, z_m)$  such that  $z_1 = x$  and  $z_m = x'$ . Its *length*, denoted by  $l(C)$ , is

$$l(C) = \sum_{j=1}^{m-1} \|z_{j+1} - z_j\|.$$

**Proposition 5.5** (Pseudo-distance, see [4, Chap. I.5, Prop. 5.19]). *The following map defines a pseudo-distance on  $\dot{X}$ ,*

$$\begin{aligned} \dot{X} \times \dot{X} &\rightarrow \mathbf{R}_+ \\ (x, x') &\rightarrow |x - x'|_{\dot{X}} = \inf \{l(C) \mid C \text{ chain between } x \text{ and } x'\}. \end{aligned}$$

By construction, the canonical maps  $X \rightarrow \dot{X}$  and  $Z \rightarrow \dot{X}$  are 1-Lipschitz. In the remainder of this section we prove that  $|\cdot|_{\dot{X}}$  is in fact a distance on  $\dot{X}$ . The next lemma is just a consequence of the triangle inequality. The proof is left to the reader.

**Lemma 5.6.** *Let  $x$  and  $x'$  be two points of  $\dot{X}$ . For all  $\eta > 0$  there is a chain  $C = (z_1, \dots, z_m)$  between them such that  $l(C) \leq |x - x'|_{\dot{X}} + \eta$  and for all  $j \in \{2, \dots, m-1\}$ ,  $z_j$  belongs to  $X$ .*

**Lemma 5.7.** *Let  $(Z, Y, \iota) \in \mathcal{Z}$ . Let  $x \in Z \setminus Y$ . Let  $d(x, Y)$  be the distance between  $x$  and  $\iota(Y)$  computed with  $|\cdot|_Z$ . For all  $x' \in \dot{X}$ , if  $|x - x'|_{\dot{X}} < d(x, Y)$  then  $x'$  belongs to  $Z$ . Moreover,  $|x - x'|_{\dot{X}} = |x - x'|_Z$ .*

*Proof.* Let  $\eta > 0$  such that  $|x - x'|_{\dot{X}} + \eta < d(x, Y)$ . A chain  $C$  between  $x$  and  $x'$  whose length is bounded above by  $|x - x'|_{\dot{X}} + \eta$  cannot go outside of  $Z$ . Therefore all its points are contained in  $Z \setminus Y$ . By triangle inequality we obtain  $|x - x'|_Z \leq l(C) \leq |x - x'|_{\dot{X}} + \eta$ . This inequality holds for all  $\eta > 0$ . Hence  $|x - x'|_Z \leq |x - x'|_{\dot{X}}$ . The other inequality follows from the definition of  $|\cdot|_{\dot{X}}$ .  $\square$

**Lemma 5.8.** *For all  $x, x' \in X$ ,  $\mu(|x - x'|_X) \leq |x - x'|_{\dot{X}} \leq |x - x'|_X$ .*

**Remark.** Recall that  $\mu$  is 1-Lipschitz and therefore continuous. The lemma shows in particular that the space  $X$  and its image in  $\dot{X}$  have the same topology.

*Proof.* The inequality  $|x - x'|_{\dot{X}} \leq |x - x'|_X$  follows directly from the definition of  $|\cdot|_{\dot{X}}$ . Let  $\eta > 0$ . By Lemma 5.6 there exists a chain  $C = (z_1, \dots, z_m)$  between  $x$  and  $x'$  whose points belong to  $X$  such that  $l(C) \leq |x - x'|_{\dot{X}} + \eta$ . Recall that for all  $j \in \{1, \dots, m-1\}$ ,  $\mu(|z_{j+1} - z_j|_X) \leq \|z_{j+1} - z_j\|$ . Using the subadditivity of  $\mu$  we get

$$\mu(|x - x'|_X) \leq \sum_{j=1}^{m-1} \mu(|z_{j+1} - z_j|_X) \leq \sum_{j=1}^{m-1} \|z_{j+1} - z_j\| = l(C).$$

Thus for all  $\eta > 0$ ,  $\mu(|x - x'|_X) \leq |x - x'|_{\dot{X}} + \eta$ , which gives the other inequality.  $\square$

**Proposition 5.9.** *For all  $x, x' \in \dot{X}$ ,  $|x - x'|_{\dot{X}} = 0$  if and only if  $x = x'$ . In particular,  $|\cdot|_{\dot{X}}$  is a distance on  $\dot{X}$ .*

*Proof.* Suppose that  $|x - x'|_{\dot{X}} = 0$ . We distinguish two cases.

- (i) Assume that there exists  $(Z, Y, \iota) \in \mathcal{Z}$  such that  $x \in Z \setminus Y$ . Since  $\iota(Y)$  is closed in  $Z$  the distance  $d(x, Y)$  is positive. Lemma 5.7 states that  $x'$  belongs to  $Z$  and  $|x - x'|_Z = |x - x'|_{\dot{X}} = 0$ . Thus  $x = x'$ .
- (ii) If  $x$  and  $x'$  both belong to  $X$  then Lemma 5.8 gives  $\mu(|x - x'|_X) \leq |x - x'|_{\dot{X}} = 0$ . Using the properties of  $\mu$ , we get  $|x - x'|_X = 0$ . Hence  $x = x'$ .

The other implication of the proposition is obvious.  $\square$

**Remark.** The proof also tells that the maps  $X \rightarrow \dot{X}$  and  $Z \rightarrow \dot{X}$  are actually embeddings.

**Proposition 5.10.** *The metric  $|\cdot|_{\dot{X}}$  endows  $\dot{X}$  with a length structure.*

*Proof.* By assumption the spaces  $X$  and  $Z$  are endowed with a length structure. Thus  $\dot{X}$  is obtained by attaching together length spaces. Since  $|\cdot|_{\dot{X}}$  is a metric,  $\dot{X}$  endowed with  $|\cdot|_{\dot{X}}$  is a length space (see [4, Chap. I.5, Lemma 5.20]).  $\square$

## 5.2 Uniform approximation of the distance

To study the curvature of  $\dot{X}$  we should understand how the cone-off construction behaves with respect to ultra-limits. To that end, we need to approximate the distance between two points of  $\dot{X}$  by a chain such that the number of points involved in this chain only depends on the error and not on the space  $X$  or the collection  $\mathcal{Z}$ . More precisely we prove the following result.

**Proposition 5.11.** *Let  $\varepsilon > 0$ . There exists  $M \geq 0$  which only depends on  $\varepsilon$  and  $a$  with the following property. Let  $x$  and  $x'$  be two points of  $X$  and  $C$  a chain between them whose points all belong to  $X$ . There is a subchain  $C'$  of  $C$  joining  $x$  and  $x'$ , which does not contain more than  $M(l(C) + 1)$  points, such that  $l(C') \leq (1 + \varepsilon)l(C) + \varepsilon$ .*

*Proof.* We write  $C = (z_1, \dots, z_m)$  for the chain between  $x$  and  $x'$ . Let  $\eta > 0$ . We define a subchain of  $C$  denoted by  $C_\eta = (z_{j_1}, \dots, z_{j_n})$  as follows.

- Put  $j_1 = 1$ .
- Let  $k \geq 1$  such that  $j_k < m$ . If  $|z_{j_{k+1}} - z_{j_k}|_X > 2\eta$ , then we put  $j_{k+1} = j_k + 1$ , otherwise  $j_{k+1}$  is the largest index  $j \in \{j_k + 1, \dots, m\}$  such that  $|z_j - z_{j_k}|_X \leq 2\eta$ .
- The process stops when  $j_k = m$ .

The chain  $C_\eta$  joins  $x$  and  $x'$ . We denote by  $n$  its number of points.

**Lemma 5.12.** *The lengths of the chains  $C$  and  $C_\eta$  satisfy  $l(C_\eta) \leq l(C) + 8an\eta^3$ .*

*Proof.* Let  $k \in \{1, \dots, n-1\}$ . For all  $t \in \mathbf{R}_+$ ,  $\mu(t) \geq t - at^3$ . By Lemma 5.8, we get

$$\sum_{l=j_k}^{j_{k+1}-1} \|z_{l+1} - z_l\| \geq |z_{j_{k+1}} - z_{j_k}|_{\dot{X}} \geq \mu(|z_{j_{k+1}} - z_{j_k}|_X) \geq \|z_{j_{k+1}} - z_{j_k}\| - a|z_{j_{k+1}} - z_{j_k}|_X^3.$$

If  $|z_{j_{k+1}} - z_{j_k}|_X \leq 2\eta$ , then

$$\sum_{l=j_k}^{j_{k+1}-1} \|z_{l+1} - z_l\| \geq \|z_{j_{k+1}} - z_{j_k}\| - 8a\eta^3.$$

If  $|z_{j_{k+1}} - z_{j_k}|_X > \eta$ , then by definition  $j_{k+1} = j_k + 1$ . Thus the last inequality holds. By summing over  $k$  we obtain

$$l(C) = \sum_{l=1}^{m-1} \|z_{l+1} - z_l\| \geq \sum_{k=1}^{n-1} \|z_{j_{k+1}} - z_{j_k}\| - 8an\eta^3 = l(C_\eta) - 8an\eta^3. \quad \square$$

**Lemma 5.13.** *If  $\eta < \sqrt{1/10a}$ , then the number of points of  $C_\eta$  is bounded as follows*

$$n \leq 4 \left( \frac{l(C)}{\eta} + 1 \right).$$

*Proof.* Let  $k$  be an integer of  $\{1, \dots, n-2\}$ . The distances  $|z_{j_{k+1}} - z_{j_k}|_X$  and  $|z_{j_{k+2}} - z_{j_{k+1}}|_X$  cannot be both bounded above by  $\eta$ . Otherwise  $j_{k+1}$  would not be the largest index  $j \in \{j_k + 1, \dots, m\}$  such that  $|z_j - z_{j_k}|_X \leq 2\eta$ . Consequently,

$$\|z_{j_{k+2}} - z_{j_{k+1}}\| + \|z_{j_{k+1}} - z_{j_k}\| \geq \mu(\eta) \geq \eta - a\eta^3.$$

By summing over  $k$  one gets

$$\left\lfloor \frac{n-1}{2} \right\rfloor (\eta - a\eta^3) \leq l(C_\eta) \leq l(C) + 8a\eta^3.$$

A small computation leads to

$$n(1 - 5a\eta^2) \leq 2 \frac{l(C)}{\eta} + 3 - 3a\eta^2.$$

Recall that  $\eta < \sqrt{1/10a}$ . Consequently,  $n \leq 4(l(C)/\eta + 1)$ .  $\square$

*End of the proof of Proposition 5.11.* Combining Lemmas 5.12 and 5.13 yields

$$l(C_\eta) \leq (1 + 32a\eta^2) l(C) + 32a\eta^3.$$

If one takes  $\eta$  small enough then  $l(C_\eta) \leq (1 + \varepsilon) l(C) + \varepsilon$ . Note that  $\eta$  only depends on  $a$  and  $\varepsilon$ . Moreover, the number of points of  $C_\eta$  is bounded above by  $4(l(C)/\eta + 1)$ .  $\square$

**Corollary 5.14.** *Let  $\varepsilon > 0$ . There is a constant  $M$  which only depends on  $\varepsilon$  and  $a$  with the following property. For all  $x, x' \in \dot{X}$ , there exists a chain  $C$  between  $x$  and  $x'$  which does not contain more than  $M(|x - x'|_{\dot{X}} + 1)$  points and such that  $l(C) \leq (1 + \varepsilon)|x - x'|_{\dot{X}} + \varepsilon$ .*

*Proof.* By Lemma 5.6, there exists a chain  $C = (z_1, \dots, z_m)$  between  $x$  and  $x'$  such that  $l(C) \leq |x - x'|_{\dot{X}} + \varepsilon/2$  and for all  $j \in \{2, \dots, m-1\}$ ,  $z_j \in X$ . We now apply Proposition 5.11 to the chain  $C_1 = (z_2, \dots, z_{m-1})$ . There exists a constant  $M$  which only depends on  $\varepsilon$  and  $a$  and a subchain  $C_2$  of  $C_1$  satisfying the followings. The chains  $C_1$  and  $C_2$  have the same extremities. The number of points of  $C_2$  is bounded above by  $M(l(C_1) + 1)$ . Moreover,  $l(C_2) \leq (1 + \varepsilon)l(C_1) + \varepsilon/2$ . We extend  $C_2$  by adding  $z_1$  at the beginning and  $z_m$  at the end. The number of points of  $C'$  is bounded above by  $M(|x - x'|_{\dot{X}} + \varepsilon/2 + 1) + 2$ . Its length satisfies

$$l(C') \leq (1 + \varepsilon)l(C) + \varepsilon/2 \leq (1 + \varepsilon)|x - x'|_{\dot{X}} + \varepsilon. \quad \square$$

### 5.3 Ultra-limit and cone-off

In this section we study the behavior of  $\dot{X}$  under ultra-limits. The data that we consider are the followings. Let  $a > 0$  and  $\omega$  be a non-principal ultra-filter. For every  $n \in \mathbf{N}$  we choose an  $a$ -comparison map  $\mu_n$ , a pointed length space  $(X_n, x_n^0)$  and a  $(\mu_n, X_n)$ -family  $\mathcal{Z}_n$ .

**Definition 5.15.** We say that the sequence  $(\mathcal{Z}_n)$  is  $\omega$ -controlled if the following holds. For every sequence of triples  $(Z_n, Y_n, \iota_n) \in \prod_{n \in \mathbf{N}} \mathcal{Z}_n$ , for every  $(y_n), (y'_n) \in \prod_{n \in \mathbf{N}} Y_n$ ,

$$\lim_{\omega} \mu_n(|y_n - y'_n|_{X_n}) = \lim_{\omega} |\iota_n(y_n) - \iota_n(y'_n)|_{Z_n}. \quad (13)$$

Since  $\mathcal{Z}_n$  is a  $(\mu_n, X_n)$  family we know that the distortion between  $X_n$  and the elements of  $\mathcal{Z}_n$  is controlled from below by  $\mu_n$  (see Definition 5.2). This definition says that, at the limit, the distortion is exactly given by  $\mu_n$ . For the remainder of this section, we assume that the distortion of  $(\mathcal{Z}_n)$  is  $\omega$ -controlled. Our goal is to study the space  $\lim_{\omega} \dot{X}_n(\mathcal{Z}_n)$ . Before, we define several objects. The metric space  $\lim_{\omega} (X_n, x_n^0)$  is denoted by  $X$ . The map  $\mu : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  is defined by  $\mu(t) = \lim_{\omega} \mu_n(t)$ . It is also an  $a$ -comparison-map.

For every  $n \in \mathbf{N}$ , for every  $(Z_n, Y_n, \iota_n) \in \mathcal{Z}_n$  we choose a point  $y_n^0 \in Y_n$  such that  $|x_n^0 - y_n^0|$  is at most  $d(x_n^0, Y_n) + 1$  (the distances here are measured with the metric of  $X_n$ ). This point exists since  $Y_n$  is non-empty. Let  $(Z_n, Y_n, \iota_n) \in \prod_{n \in \mathbf{N}} \mathcal{Z}_n$  be a sequence of triples. We define two limit spaces

- $Z = \lim_{\omega} (Z_n, \iota_n(y_n^0))$ ,
- $Y = \lim_{\omega} Y_n$  which is a (possibly empty) subset of  $X = \lim_{\omega} X_n$ .

It follows from (13) that the map  $\iota : Y \rightarrow Z$  given by  $\iota(\lim_{\omega} y_n) = \lim_{\omega} \iota_n(y_n)$  is well defined. We write then  $(Z, Y, \iota) = \lim_{\omega} (Z_n, Y_n, \iota_n)$ . This triple also satisfies the following property. For all  $y, y' \in Y$

$$|\iota(y) - \iota(y')|_Z = \mu(|y - y'|_X) \leq |y - y'|_X. \quad (14)$$

Note that the set  $Y = \lim_{\omega} Y_n$  is non-empty if and only if  $(|x_n^0 - y_n^0|)$  is  $\omega$ -eb. We write  $\prod_{\omega} \mathcal{Z}_n$  for the set of sequences of triples satisfying this last condition.

We endow the product  $\prod_{n \in \mathbf{N}} \mathcal{Z}_n$  with the following equivalence relation. Given two sequences of triples,  $(Z_n, Y_n, \iota_n) \sim (Z'_n, Y'_n, \iota'_n)$  if  $(Z_n, Y_n, \iota_n) = (Z'_n, Y'_n, \iota'_n)$   $\omega$ -as. In particular they define the same limit triple  $(Z, Y, \iota) = \lim_{\omega} (Z_n, Y_n, \iota_n) = \lim_{\omega} (Z'_n, Y'_n, \iota'_n)$ . We can now define  $\mathcal{Z}$  to be the set of triples  $(Z, Y, \iota) = \lim_{\omega} (Z_n, Y_n, \iota_n)$  where  $(Z_n, Y_n, \iota_n) \in \prod_{\omega} \mathcal{Z}_n / \sim$ . It follows from (14), that  $\mathcal{Z}$  is a  $(\mu, X)$ -family. It allow us to look at the cone-off  $\dot{X}(\mathcal{Z})$  over  $X$  relatively to  $\mathcal{Z}$ .

Our goal is to compare  $\lim_{\omega} \dot{X}_n(\mathcal{Z}_n)$  with the metric space  $\dot{X}(\mathcal{Z})$ . To that end we define the following maps (the second kind of maps are defined for every  $(Z, Y, \iota)$  in  $\mathcal{Z}$ ).

$$\begin{array}{lll} \psi_X : & X & \rightarrow \lim_{\omega} \dot{X}_n \\ \lim_{\omega} x_n & \rightarrow & \lim_{\omega} x_n \end{array} \quad \begin{array}{lll} \psi_Z : & Z & \rightarrow \lim_{\omega} \dot{X}_n \\ \lim_{\omega} x_n & \rightarrow & \lim_{\omega} x_n \end{array}$$

Recall that for every  $n \in \mathbf{N}$ , the embedding  $X_n \hookrightarrow \dot{X}_n$  (respectively  $Z_n \hookrightarrow \dot{X}_n$ ) is 1-Lipschitz. Thus the maps  $\psi$  and  $\psi_Z$  are well-defined and 1-Lipschitz. Moreover, for all  $(Z, Y, \iota) \in \mathcal{Z}$ , for all  $y \in Y$ ,  $\psi_Z \circ \iota(y) = \psi_X(y)$ . Consequently, they induce a map  $\psi : \dot{X} \rightarrow \lim_{\omega} \dot{X}_n$  whose restriction to  $X$  (respectively  $Z$ ) is  $\psi_X$  (respectively  $\psi_Z$ ).

The map  $\psi$  cannot be an isometry. Indeed the space  $\lim_{\omega} \dot{X}_n$  is much larger than  $\dot{X}$ . Imagine for instance that the spaces  $Z_n$  that we attach are uniformly bounded. This will be the case later: we will take for  $Z_n$  some cones with a fixed radius. One can find a sequence of point  $(x_n)$  of  $\prod_{n \in \mathbf{N}} X_n$  such that the distance  $|x_n^0 - x_n|$  is bounded in  $\dot{X}_n$  but not in  $X_n$ . Therefore this sequence defines a point of  $\lim_{\omega} \dot{X}_n$  which does not correspond to a point of  $\dot{X}$ . Nevertheless  $\psi$  restricted to the neighborhood of  $X$  in  $\dot{X}$  induces a local isometry. More precisely we are going to prove the following result.

**Proposition 5.16.** *Let  $x$  be a point of  $X$ . Let  $t \in \mathbf{R}_+$ . The map  $\dot{\psi}$  induces an isometry from  $B(x, \mu(t)/2)$  onto  $B(\dot{\psi}(x), \mu(t)/2)$ .*

The rest of this section is dedicated to the proof of Proposition 5.16. We begin by establishing that  $\dot{\psi}$  is 1-lipschitz, and then that it induces a local isometry.

**Lemma 5.17.** *If  $x$  and  $x'$  are two points of  $\dot{X}$  then  $\|x - x'\| \geq |\dot{\psi}(x) - \dot{\psi}(x')|$ .*

*Proof.* If  $\|x - x'\| = +\infty$  there is nothing to prove. Thus we may assume that  $x$  and  $x'$  both belong either to  $X$  or to one of the  $Z$  where  $(Z, Y, \iota) \in \mathcal{Z}$ . Suppose that  $x, x' \in X$ . We can write  $x = \lim_{\omega} x_n$  and  $x' = \lim_{\omega} x'_n$  where  $x_n, x'_n \in X_n$ . By construction of  $\dot{X}_n$  we have for all  $n \in \mathbf{N}$ ,  $|x_n - x'_n|_{X_n} \geq |x_n - x'_n|_{\dot{X}_n}$ . After taking the  $\omega$ -limit we get  $|x - x'|_X \geq |\psi_X(x) - \psi_X(x')|$ . Consider now  $(Z, Y, \iota) = \lim_{\omega} (Z_n, Y_n, \iota_n) \in \mathcal{Z}$ . If  $x$  and  $x'$  both belong to  $Z$  we prove in the same way that  $|x - x'|_Z \geq |\psi_Z(x) - \psi_Z(x')|$ . It follows from the definition of  $\|\cdot\|$  that  $\|x - x'\| \geq |\dot{\psi}(x) - \dot{\psi}(x')|$ .  $\square$

**Proposition 5.18.** *The map  $\dot{\psi}$  is 1-lipschitz.*

*Proof.* Let  $x$  and  $x'$  be two points of  $\dot{X}$ . Let  $C = (z_0, \dots, z_m)$  be a chain of points joining  $x$  and  $x'$ . By Lemma 5.17, we have

$$|\dot{\psi}(x) - \dot{\psi}(x')| \leq \sum_{j=0}^{m-1} |\dot{\psi}(z_{j+1}) - \dot{\psi}(z_j)| \leq \sum_{j=0}^{m-1} \|z_{j+1} - z_j\| = l(C).$$

This inequality holds for all chains joining  $x$  and  $x'$ , thus  $|\dot{\psi}(x) - \dot{\psi}(x')| \leq \|x - x'\|_{\dot{X}}$ .  $\square$

**Lemma 5.19.** *Let  $(x_n)$  and  $(x'_n)$  be two sequences of  $\prod_{\omega} \dot{X}_n$  such that  $\lim_{\omega} |x_n - x'_n|_{\dot{X}_n} = 0$ .*

(i) *Assume that  $x_n, x'_n \in X_n$   $\omega$ -as, then  $\lim_{\omega} |x_n - x'_n|_{X_n} = 0$ .*

(ii) *Assume that there exists a sequence of triples  $(Z_n, Y_n, \iota_n) \in \prod_{n \in \mathbf{N}} \mathcal{Z}_n$  such that  $x_n, x'_n \in Z_n$   $\omega$ -as, then  $\lim_{\omega} |x_n - x'_n|_{Z_n} = 0$ .*

**Remark.** In this lemma the sequences  $(x_n)$  and  $(x'_n)$  do not necessarily define points of  $X$  or  $Z \in \mathcal{Z}$ . In particular we do not assume that  $(Z_n, Y_n, \iota_n)$  belongs to  $\prod_{\omega} \mathcal{Z}_n$ . In other words  $Y = \lim_{\omega} Y_n$  may be empty.

*Proof.* Assume that  $x_n, x'_n$  belong to  $X_n$   $\omega$ -as. By Lemma 5.8,

$$|x_n - x'_n|_{X_n} - a|x_n - x'_n|_{X_n}^3 \leq \mu_n(|x_n - x'_n|_{X_n}) \leq |x_n - x'_n|_{\dot{X}_n} \omega\text{-as}.$$

In particular,  $\lim_{\omega} |x_n - x'_n|_{X_n} = 0$ .



Assume now that there is  $(Z_n, Y_n, \iota_n) \in \prod_{n \in \mathbf{N}} \mathcal{Z}_n$  such that  $x_n, x'_n \in Z_n$   $\omega$ -as. By Lemma 5.6, for every  $n \in \mathbf{N}$ , there is a chain  $C_n$  between  $x_n$  and  $x'_n$  such that  $\lim_{\omega} l(C_n) = 0$  and every point of  $C_n$  distinct from  $x_n$  and  $x'_n$  belongs to  $X_n$ . By doubling if necessary the first and the last point of  $C_n$  we can assume the following. The second point  $y_n$  and the last point but one  $y'_n$  of  $C_n$  belong to  $Y_n$ . Moreover,  $\|x_n - y_n\| = |x_n - y_n|_{Z_n}$  and  $\|x'_n - y'_n\| = |x'_n - y'_n|_{Z_n}$ . It follows that

$$l(C_n) \geq |x_n - y_n|_{Z_n} + |y_n - y'_n|_{\dot{X}_n} + |y'_n - x'_n|_{Z_n}.$$

In particular the following quantities  $|x_n - y_n|_{Z_n}$ ,  $|y_n - y'_n|_{\dot{X}_n}$  and  $|y'_n - x'_n|_{Z_n}$  converge to zero. The triangle inequality leads to  $\lim_{\omega} |x_n - x'_n|_{Z_n} = \lim_{\omega} |y_n - y'_n|_{Z_n}$ . However  $y_n, y'_n$  belong to  $Y_n$   $\omega$ -as. It follows from the previous point that  $\lim_{\omega} |y_n - y'_n|_{X_n} = 0$ . Combined with (13) it gives

$$\lim_{\omega} |x_n - x'_n|_{Z_n} = \lim_{\omega} |y_n - y'_n|_{Z_n} = \lim_{\omega} \mu_n(|y_n - y'_n|_{X_n}) = \mu\left(\lim_{\omega} |y_n - y'_n|_{X_n}\right) = 0. \quad \square$$

**Lemma 5.20.** *Let  $x$  and  $x'$  be two points of  $X$ . Let  $(z_n)$  and  $(z'_n)$  be two sequences which belong to  $\prod_{\omega} \dot{X}_n$  such that  $\psi_X(x) = \lim_{\omega} z_n$  and  $\psi_X(x') = \lim_{\omega} z'_n$ . Then  $\lim_{\omega} \|z_n - z'_n\| \geq \|x - x'\|$ .*

*Proof.* We can write that  $x = \lim_{\omega} x_n$  and  $x' = \lim_{\omega} x'_n$  where  $x_n, x'_n \in X_n$ . By definition  $\psi_X(x)$  and  $\psi_X(x')$  are respectively  $\lim_{\omega} x_n$  and  $\lim_{\omega} x'_n$  but seen as points of  $\lim_{\omega} \dot{X}_n$ , thus  $\lim_{\omega} |x_n - z_n|_{\dot{X}_n} = 0$  and  $\lim_{\omega} |x'_n - z'_n|_{\dot{X}_n} = 0$ . We distinguish two cases.

**First case.** Assume that  $z_n$  and  $z'_n$  both belong to  $X_n$   $\omega$ -as. By Lemma 5.19,  $\lim_{\omega} |x_n - z_n|_{X_n} = 0$  and  $\lim_{\omega} |x'_n - z'_n|_{X_n} = 0$ . Using the triangle inequality we get

$$\lim_{\omega} |z_n - z'_n|_{X_n} = \lim_{\omega} |x_n - x'_n|_{X_n} = |x - x'|_X \geq \|x - x'\|.$$

**Second case.** Assume that there exists  $(Z_n, Y_n, \iota_n) \in \prod_{n \in \mathbf{N}} \mathcal{Z}_n$  such that  $z_n, z'_n \in Z_n$   $\omega$ -as. We write  $(Z, Y, \iota)$  for  $\lim_{\omega} (Z_n, Y_n, \iota_n)$ . It follows from Lemma 5.19 and the definition of the metric on  $\dot{X}_n$  that there exists a sequence  $(y_n)$  such that  $y_n \in Y_n$   $\omega$ -as and

$$\lim_{\omega} (|x_n - y_n|_{X_n} + |y_n - z_n|_{Z_n}) = 0.$$

Hence  $\lim_{\omega} y_n$  defines a point of  $X$  which equals  $x$ . In particular  $Y$  is non-empty, which means that the sequence  $(Z_n, Y_n, \iota_n)$  belongs in fact to  $\prod_{\omega} \mathcal{Z}_n$ . We construct an analogue sequence  $(y'_n)$  for  $x'$ . By triangle inequality

$$\lim_{\omega} |z_n - z'_n|_{Z_n} = \lim_{\omega} |y_n - y'_n|_{Z_n} = |\iota(x) - \iota(x')|_Z \geq \|x - x'\|.$$

Note that the  $\omega$ -limit and the infimum that defines  $\|\cdot\|$  can be swapped. Consequently,  $\lim_{\omega} \|z_n - z'_n\| \geq \|x - x'\|$ .  $\square$

**Lemma 5.21.** *Let  $(Z, Y, \iota) = \lim_{\omega} (Z_n, Y_n, \iota_n) \in \mathcal{Z}$ . Let  $x$  and  $x'$  be two points of  $Z$ . Let  $(z_n)$  and  $(z'_n)$  be two sequences which belong to  $\prod_{\omega} \dot{X}_n$  such that  $\psi_Z(x) = \lim_{\omega} z_n$  and  $\psi_Z(x') = \lim_{\omega} z'_n$ . Then  $\lim_{\omega} \|z_n - z'_n\| \geq \|x - x'\|$ .*

*Proof.* The proof is essentially the same as for the previous lemma. By (14)  $\|x - x'\| = |x - x'|_Z$ . We can write that  $x = \lim_{\omega} x_n$  and  $x' = \lim_{\omega} x'_n$  where  $x_n, x'_n \in Z_n$ . By definition  $\psi_Z(x)$  and  $\psi_Z(x')$  are respectively  $\lim_{\omega} x_n$  and  $\lim_{\omega} x'_n$  but seen as points of  $\lim_{\omega} \dot{X}_n$ , thus  $\lim_{\omega} |x_n - z_n|_{\dot{X}_n} = 0$  and  $\lim_{\omega} |x'_n - z'_n|_{\dot{X}_n} = 0$ . We distinguish three cases.

**First case.** Assume that  $z_n$  and  $z'_n$  both belong to  $X_n$   $\omega$ -as. It follows from Lemma 5.19 and the definition of the metric on  $\dot{X}_n$  that there exists a sequence  $(y_n)$  such that  $y_n \in Y_n$   $\omega$ -as and

$$\lim_{\omega} (|x_n - y_n|_{Z_n} + |y_n - z_n|_{X_n}) = 0.$$

We construct an analogue sequence  $(y'_n)$  for  $x'$ . The triangle inequality gives

$$\lim_{\omega} |z_n - z'_n|_{X_n} = \lim_{\omega} |y_n - y'_n|_{X_n} \geq \mu \left( \lim_{\omega} |y_n - y'_n|_{X_n} \right).$$

It follows from (13) that

$$\lim_{\omega} |z_n - z'_n|_{X_n} \geq \lim_{\omega} |y_n - y'_n|_{Z_n} = \lim_{\omega} |x_n - x'_n|_{Z_n} = \|x - x'\|.$$

**Second case.** Assume that  $z_n, z'_n \in Z_n$   $\omega$ -as. It follows from Lemma 5.19 that the limits  $\lim_{\omega} |x_n - z_n|_{Z_n}$  and  $\lim_{\omega} |x'_n - z'_n|_{Z_n}$  equal zero. By the triangle inequality,  $\lim_{\omega} |z_n - z'_n|_{Z_n} = \lim_{\omega} |x_n - x'_n|_{Z_n} = \|x - x'\|$ .

**Third case.** Assume that there exists  $(Z'_n, Y'_n, t'_n) \in \prod_{n \in \mathbf{N}} \mathcal{Z}_n$  whose limit  $(Z', Y', t')$  is distinct from  $(Z, Y, t)$  and such that  $z_n, z'_n \in Z'_n$   $\omega$ -as. It follows from Lemma 5.19 and the definition of the metric on  $\dot{X}_n$  that there exist two sequences  $(y_n)$  and  $(t_n)$  such that  $y_n \in Y_n$ ,  $t_n \in Y'_n$   $\omega$ -as and

$$\lim_{\omega} (|x_n - y_n|_{Z_n} + |y_n - t_n|_{X_n} + |t_n - z_n|_{Z'_n}) = 0.$$

In the same way we define two sequences  $(y'_n)$  and  $(t'_n)$  for  $x'$ . Using the triangle inequality and (13) we have

$$\begin{aligned} \lim_{\omega} |z_n - z'_n|_{Z'_n} &= \lim_{\omega} |t_n - t'_n|_{Z'_n} = \lim_{\omega} \mu_n (|t_n - t'_n|_{X_n}), \\ \lim_{\omega} |x_n - x'_n|_{Z_n} &= \lim_{\omega} |y_n - y'_n|_{Z_n} = \lim_{\omega} \mu_n (|y_n - y'_n|_{X_n}). \end{aligned}$$

Nevertheless  $\mu_n$  is 1-Lipschitz thus

$$\begin{aligned} |\mu_n (|t_n - t'_n|_{X_n}) - \mu_n (|y_n - y'_n|_{X_n})| &\leq ||t_n - t'_n|_{X_n} - |y_n - y'_n|_{X_n}| \\ &\leq |y_n - t_n|_{X_n} + |y'_n - t'_n|_{X_n} \end{aligned}$$

Consequently,  $\lim_{\omega} |z_n - z'_n|_{Z'_n} = \lim_{\omega} |x_n - x'_n|_{Z_n} = \|x - x'\|$ .

Note that the  $\omega$ -limit and the infimum that defines  $\| \cdot \|$  can be swapped. It follows that  $\lim_{\omega} \|z_n - z'_n\| \geq \|x - x'\|$ .  $\square$

**Lemma 5.22.** Let  $C = (z^0, \dots, z^m)$  be a chain between two points of  $\dot{X}$ . For all  $j \in \{0, \dots, m\}$  we consider a sequence  $(z_n^j) \in \prod_{\omega} \dot{X}_n$  such that  $\dot{\psi}(z^j) = \lim_{\omega} z_n^j$ . For all  $n \in \mathbf{N}$ , we define a chain of  $\dot{X}_n$  as follows:  $C_n = (z_n^0, \dots, z_n^m)$ . Then  $l(C) \leq \lim_{\omega} l(C_n)$ .

*Proof.* It follows directly from Lemmas 5.20 and 5.21.  $\square$

**Lemma 5.23.** Let  $x = \lim_{\omega} x_n$  be a point of  $X$ . Let  $t \in \mathbf{R}_+$ . Let  $z$  be a point of  $B(\dot{\psi}(x), \mu(t))$ . There exists  $x' \in \dot{X}$  such that  $\dot{\psi}(x') = z$ .

*Proof.* Recall that  $B(\dot{\psi}(x), \mu(t))$  is a subset of  $\lim_{\omega} \dot{X}_n$ . Thus we can write  $z = \lim_{\omega} z_n$ , where  $z_n \in \dot{X}_n$ . Since  $z$  belongs to  $B(\dot{\psi}(x), \mu(t))$ ,  $\lim_{\omega} |x_n - z_n|_{\dot{X}_n} < \mu(t)$ . We distinguish two cases.

**First case.** Assume that  $z_n$  belongs to  $X_n$   $\omega$ -as. By Lemma 5.8

$$\lim_{\omega} \mu_n (|x_n - z_n|_{X_n}) \leq \lim_{\omega} |x_n - z_n|_{\dot{X}_n} < \mu(t) = \lim_{\omega} \mu_n(t).$$

Since  $\mu_n$  is non-decreasing,  $|x_n - z_n|_{X_n} < t$ ,  $\omega$ -as. Thus  $\lim_{\omega} z_n$  defines a point of  $X$ , whose image by  $\dot{\psi}$  is  $z$ .

**Second case.** Assume that there is  $(Z_n, Y_n, \iota_n) \in \prod_{n \in \mathbf{N}} \mathcal{Z}_n$  such that  $z_n \in Z_n$   $\omega$ -as. By the definition of the distance on  $\dot{X}_n$  there is a sequence  $(y_n)$  such that  $y_n \in Y_n$   $\omega$ -as and

$$\lim_{\omega} (|x_n - y_n|_{\dot{X}_n} + |y_n - z_n|_{Z_n}) \leq \lim_{\omega} |x_n - z_n|_{\dot{X}_n} < \mu(t) = \lim_{\omega} \mu_n(t).$$

As above we prove that  $y = \lim_{\omega} y_n$  is a well defined point of  $X$ . Moreover, since the sequence  $(|x_n - y_n|_{X_n})$  is  $\omega$ -eb,  $(Z, Y, \iota) = \lim_{\omega} (Z_n, Y_n, \iota_n)$  is in fact an element of  $\mathcal{Z}$ . It follows that  $\lim_{\omega} z_n$  defines a point of  $Z$  whose image by  $\psi_Z$  is  $z$ .  $\square$

**Lemma 5.24.** *Let  $x = \lim_{\omega} x_n$  be a point of  $X$ . Let  $t \in \mathbf{R}_+$ . Let  $y, y'$  be two points of  $\dot{X}$  such that  $\dot{\psi}(y), \dot{\psi}(y')$  belong to  $B(\dot{\psi}(x), \mu(t)/2)$ . Then  $|\dot{\psi}(y) - \dot{\psi}(y')| = |y - y'|$ .*

*Proof.* By assumption  $|\dot{\psi}(y) - \dot{\psi}(y')| \leq |\dot{\psi}(y) - \dot{\psi}(x)| + |\dot{\psi}(x) - \dot{\psi}(y')| < \mu(t)$ . Let  $\eta > 0$  such that  $|\dot{\psi}(y) - \dot{\psi}(y')| + \eta < \mu(t)$ . There exist two sequences  $(z_n)$  and  $(z'_n)$  of  $\prod_{\omega} \dot{X}_n$  such that  $\dot{\psi}(y) = \lim_{\omega} z_n$  and  $\dot{\psi}(y') = \lim_{\omega} z'_n$ . It follows that  $|z_n - z'_n|_{\dot{X}_n} + \eta < \mu(t)$   $\omega$ -as. According to Proposition 5.14 there exists an integer  $m$  and, for all  $n \in \mathbf{N}$ , a chain  $C_n = (z_n^0, \dots, z_n^m)$  of  $\dot{X}_n$  between  $z_n$  and  $z'_n$  such that  $l(C_n) \leq |z_n - z'_n|_{\dot{X}_n} + \eta$ . It is worth pointing out that  $m$ , the number of points of  $C_n$ , does not depend on  $n$ . Note also that for all  $j \in \{0, \dots, m\}$  the distance between  $z_n^j$  and one of the points  $z_n$  and  $z'_n$  is less than  $\mu(t)/2$ . Thus  $z^j = \lim_{\omega} z_n^j$  is a well defined point of  $\lim_{\omega} \dot{X}_n$ . Moreover, its distance to either  $\dot{\psi}(y)$  or  $\dot{\psi}(y')$  is less than  $\mu(t)/2$ . Consequently,  $z^j$  belongs to  $B(\dot{\psi}(x), \mu(t))$ . By Lemma 5.23 there is a point  $y^j \in \dot{X}$  such that  $\dot{\psi}(y^j) = z^j$ . We choose  $y^0 = y$  and  $y^m = y'$ . Hence  $C = (y^0, \dots, y^m)$  is a chain of  $\dot{X}$  between  $y$  and  $y'$ . According to Lemma 5.22 its length satisfies

$$l(C) \leq \lim_{\omega} l(C_n) \leq \lim_{\omega} |z_n - z'_n|_{\dot{X}_n} + \eta \leq |\dot{\psi}(y) - \dot{\psi}(y')| + \eta.$$

Therefore  $|y - y'|_{\dot{X}} \leq |\dot{\psi}(y) - \dot{\psi}(y')| + \eta$ . This inequality holds for all  $\eta > 0$ . Thus  $|y - y'|_{\dot{X}} \leq |\dot{\psi}(y) - \dot{\psi}(y')|$ . Recall that  $\dot{\psi}$  is 1-lipschitz (Lemma 5.18). This gives  $|y - y'|_{\dot{X}} = |\dot{\psi}(y) - \dot{\psi}(y')|$ .  $\square$

*Proof of Proposition 5.16.* Let  $y$  and  $y'$  be two points of  $B(x, \mu(t)/2)$ . Since  $\dot{\psi}$  is 1-lipschitz,  $\dot{\psi}(y)$  and  $\dot{\psi}(y')$  belong to the ball  $B(\dot{\psi}(x), \mu(t)/2)$ . By Lemma 5.24,  $|\dot{\psi}(y) - \dot{\psi}(y')| = |y - y'|$ . Thus  $\dot{\psi}$  preserves the distances. It remains to prove that  $\dot{\psi}$  is onto. Let  $z$  be a point of  $B(\dot{\psi}(x), \mu(t)/2)$ . According to Lemma 5.23 there is  $y \in \dot{X}$  such that  $\dot{\psi}(y) = z$ . We should prove show that  $y$  belong to  $B(x, \mu(t)/2)$ . By construction  $\dot{\psi}(x)$  and  $\dot{\psi}(y)$  are two points of  $B(\dot{\psi}(x), \mu(t)/2)$ . It follows from Lemma 5.24 that  $|x - y| = |\dot{\psi}(x) - \dot{\psi}(y)| < \mu(t)/2$ .  $\square$

## 5.4 Hyperbolicity of the cone-off

Let  $a > 0$  and  $\mu$  be an  $a$ -comparison map. In section we study the curvature of the space  $\dot{X}(\mathcal{Z})$  when the base  $X$  is a hyperbolic length space and  $\mathcal{Z}$  a  $(\mu, X)$ -family such that for every  $(Z, Y, \iota) \in \mathcal{Z}$ ,  $Z$  is hyperbolic. We start with the special case of tree graded space.

**Definition 5.25.** Let  $X$  be a metric length space. Let  $\mathcal{P}$  be a collection of closed path-connected subsets (called *pieces*). We say that  $X$  is *tree graded* with respect to  $\mathcal{P}$  if the following holds

- (i) Every two different pieces have at most one common point.
- (ii) Every simple loop is contained in contained in one piece.

**Remark.** The pieces in this definition have no relation the the pieces of the usual small cancellation theory! For more details about tree-graded spaces we refer the reader to Drutu:2005tz.

**Proposition 5.26.** Let  $\delta > 0$ . Let  $a > 0$  and  $\mu$  be an  $a$ -comparison map. Let  $X$  be a tree-graded space with respect to  $\mathcal{P}$ . Let  $\mathcal{Z}$  a  $(\mu, X)$ -family. We assume that there is a one-to-one correspondence between  $\mathcal{P}$  and the the collection of subsets  $Y$  belonging to a triple  $(Z, Y, \iota)$  of  $\mathcal{Z}$ . Then  $\dot{X}(\mathcal{Z})$  is  $\delta$ -hyperbolic as well.

**Remark.** In particular if  $X$  is an  $\mathbf{R}$ -tree and the  $Y$ 's are subtrees sharing at most one point then the proposition applies.

*Proof.* By construction the space  $\dot{X}(\mathcal{Z})$  is a tree-graded space with respect to the collection  $\{Z \mid (Z, Y, \iota) \in \mathcal{Z}\}$ . Every piece in this new structure is  $\delta$ -hyperbolic. However, if one glues together two  $\delta$ -hyperbolic spaces sharing exactly one point one gets a  $\delta$ -hyperbolic space. Therefore  $\dot{X}(\mathcal{Z})$  is  $\delta$ -hyperbolic.  $\square$

The next result is a small “perturbation” of the previous one. In particular  $X$  is no more a tree but a  $\delta_0$ -hyperbolic space. Moreover, we assume that the  $Y$ 's are  $2\delta_0$ -quasi-convex and allow them to have a small overlap. To measure that overlap we introduce the parameter  $\Delta$ .

$$\Delta(\mathcal{Z}) = \sup \left\{ \text{diam} \left( Y_1^{+5\delta_0} \cap Y_2^{+5\delta_0} \right) \mid (Z_1, Y_1, \iota_1) \neq (Z_2, Y_2, \iota_2) \in \mathcal{Z} \right\}.$$

**Proposition 5.27.** Let  $\delta \geq 0$ . Let  $a, \eta, t > 0$ . There exist positive constants  $\delta_0 = \delta_0(\delta, \eta, a, t)$  and  $\Delta_0 = \Delta_0(\delta, \eta, a, t)$  satisfying the following property. Let  $\mu$  be an  $a$ -comparison map. Let  $X$  be a  $\delta_0$ -hyperbolic length space and  $\mathcal{Z}$  be a  $(\mu, X)$ -family such that for every  $(Z, Y, \iota) \in \mathcal{Z}$ ,  $Z$  is  $\delta$ -hyperbolic length space,  $Y$  is a  $2\delta_0$ -quasi-convex subset of  $X$  and for all  $y, y' \in Y$

$$\mu(|y - y'|_X) \leq |\iota(y) - \iota(y')|_Z \leq \mu(|y - y'|_X) + 8\delta_0.$$

If  $\Delta(\mathcal{Z}) \leq \Delta_0$  then every ball of radius  $\mu(t)/8$  of  $\dot{X}(\mathcal{Z})$  is  $(\delta + \eta)$ -hyperbolic.

*Proof.* We argue by contradiction. Assume that the proposition is false. For every  $n \in \mathbf{N}$  one can find

- (i) an  $a$ -comparison map  $\mu_n : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ ,
- (ii) a geodesic,  $\delta_n$ -hyperbolic length space  $X_n$  where  $\delta_n = o(1)$ ,

- (iii) a  $(\mu_n, X_n)$ -family  $\mathcal{Z}_n$  such that for every  $(Z, Y, \iota)$  in  $\mathcal{Z}_n$ , the space  $Z$  is a  $\delta$ -hyperbolic length space,  $Y_n$  is a  $2\delta_n$ -quasi-convex subset of  $X_n$  and for all  $y, y' \in Y_n$

$$\mu(|y - y'|_{X_n}) \leq |\iota(y) - \iota(y')|_{Z_n} \leq \mu(|y - y'|_{X_n}) + 8\delta_n. \quad (15)$$

Moreover,  $\Delta(\mathcal{Z}_n) = o(1)$ ,

- (iv) a point  $x_n$  in  $\dot{X}_n(\mathcal{Z}_n)$  such that the ball  $B_n = B(x_n, \mu_n(t)/8)$  is not  $(\delta + \eta)$ -hyperbolic.

Let us fix a non-principal ultra-filter  $\omega$ . First note that  $d(x_n, X_n) < 3\mu_n(t)/8$   $\omega$ -as. Otherwise there is a sequence of triples  $(Z_n, Y_n, \iota_n) \in \prod_{n \in \mathbf{N}} \mathcal{Z}_n$  such that  $x_n \in Z_n$   $\omega$ -as. Moreover, by Lemma 5.7, the ball  $B_n$  is contained in  $Z_n$  and the metrics  $|\cdot|_{\dot{X}_n}$  and  $|\cdot|_{Z_n}$  coincide on  $B_n$ . Since  $Z_n$  is  $\delta$ -hyperbolic, so is  $B_n$ , a contradiction. We denote by  $x_n^0$  a point of  $X_n$  such that  $|x_n - x_n^0|_{\dot{X}_n} \leq 3\mu_n(t)/8$ . In particular  $B_n$  is contained in the ball  $B(x_n^0, \mu_n(t)/2)$  of  $\dot{X}_n$ .

Since  $X_n$  is a  $\delta_n$ -hyperbolic length space, the space  $X = \lim_{\omega} (X_n, x_n^0)$  is an  $\mathbf{R}$ -tree. We denote by  $x^0$  the point  $x^0 = \lim_{\omega} x_n^0$ . We define an  $a$ -comparison map  $\mu : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  given by  $\mu(s) = \lim_{\omega} \mu_n(s)$  for all  $s \in \mathbf{R}_+$ . It follows from our assumption (15) that the distortion of  $(\mathcal{Z}_n)$  is  $\omega$ -controlled. As explained in Section 5.3, we construct a  $(\mu, X)$ -family  $\mathcal{Z}$  and a map

$$\dot{\psi} : \dot{X}(\mathcal{Z}) \rightarrow \lim_{\omega} (\dot{X}_n(\mathcal{Z}_n), x_n^0).$$

According to Proposition 5.16,  $\dot{\psi}$  induces an isometry from  $B(x^0, \mu(t)/2)$  onto  $B(\dot{\psi}(x^0), \mu(t)/2)$  which is exactly  $\lim_{\omega} B(x_n^0, \mu_n(t)/2)$ .

Let  $(Z, Y, \iota) = \lim_{\omega} (Z_n, Y_n, \iota_n)$  be an element of  $\mathcal{Z}$ . By assumption for every  $n \in \mathbf{N}$ ,  $Z_n$  is  $\delta$ -hyperbolic, thus so is  $Z = \lim_{\omega} Z_n$ . On the other hand, for every  $n \in \mathbf{N}$ ,  $Y_n$  is a  $2\delta_n$ -quasi-convex subset of  $X_n$ . Thus  $Y = \lim_{\omega} Y_n$  is a subtree of  $X$ . Consider now an other triple  $(Z', Y', \iota') = \lim_{\omega} (Z'_n, Y'_n, \iota'_n)$  distinct from  $(Z, Y, \iota)$ . In particular  $(Z_n, Y_n, \iota_n) \neq (Z'_n, Y'_n, \iota'_n)$ ,  $\omega$ -as. We assumed that  $\Delta(\mathcal{Z}_n)$  tends to zero as  $n$  approaches infinity. By Proposition 2.20  $\text{diam}(Y \cap Y') = 0$ . Thus  $Y$  and  $Y'$  share at most one point. Consequently,  $\mu$ ,  $X$  and  $\mathcal{Z}$  satisfy the assumptions of Proposition 5.26. Hence  $\dot{X}(\mathcal{Z})$  is  $\delta$ -hyperbolic. It follows that  $\lim_{\omega} B_n$  is also  $\delta$ -hyperbolic. By Proposition 2.19,  $B_n$  is  $(\delta + \eta)$ -hyperbolic  $\omega$ -as, which contradicts our assumptions.  $\square$

## 6 Small cancellation theory

### 6.1 General framework

In this section  $X$  is a proper  $\delta$ -hyperbolic geodesic space endowed with a proper and co-compact action by isometries of a group  $G$ . We consider a family  $\mathcal{Q}$  of pairs  $(H, Y)$  where  $Y$  is a strongly quasi-convex subset of  $X$  and  $H$  a subgroup of  $\text{Stab}(Y)$  acting co-compactly on  $Y$ . We assume that  $G$  acts on  $\mathcal{Q}$  and that  $\mathcal{Q}/G$  is finite. The action of  $G$  on  $\mathcal{Q}$  is defined as follows. For every  $(H, Y) \in \mathcal{Q}$ , for every  $g \in G$ ,  $g(H, Y) = (gHg^{-1}, gY)$ . In some applications the spaces  $Y$ 's might not be strongly quasi-convex but simply uniformly quasi-convex (i.e., there exists  $\alpha$  such that all the  $Y$ 's are  $\alpha$ -quasi-convex). Then we can substitute  $Y$  for an appropriate neighborhood of  $Y$  (see

Lemma 2.21) to satisfy the quasi-convexity assumption. In Section 7.2 we give an example of such an operation.

We denote by  $K$  the (normal) subgroup generated by the subgroups  $H$  where  $(H, Y) \in \mathcal{Q}$ . Our goal is to understand the quotient  $\bar{G} = G/K$ . To that end we consider two parameters which respectively play the role of the length of the largest piece and the length of the smallest relation in the usual small cancellation theory.

$$\begin{aligned}\Delta(\mathcal{Q}) &= \sup \{ \text{diam}(Y_1^{+5\delta} \cap Y_2^{+5\delta}) \mid (H_1, Y_1) \neq (H_2, Y_2) \in \mathcal{Q} \} \\ T(\mathcal{Q}) &= \inf \{ [h] \mid h \in H \setminus \{1\}, (H, Y) \in \mathcal{Q} \}\end{aligned}$$

Let us recall the general strategy sketched in the introduction. First we construct a space  $\dot{X}$  by attaching on  $X$  cones of bases  $Y$ , where  $(H, Y) \in \mathcal{Q}$ . Under a small cancellation assumption on  $\Delta(\mathcal{Q})$  and  $T(\mathcal{Q})$  it turns out that  $\dot{X}$  is hyperbolic. Moreover the groups  $H$  where  $(H, Y) \in \mathcal{Q}$  define a rotation family. This allow us to apply the results of Section 3. In particular we will see that  $\bar{G}$  acts by isometries on the space  $\bar{X} = \dot{X}/K$  which is also hyperbolic. In the next sections we will study the properties of the action of  $\bar{G}$  on  $\bar{X}$ . In particular we provide estimates for the invariants  $A(\bar{G}, \bar{X})$  and  $r_{inj}(\bar{G}, \bar{X})$ .

**Notation.** In this section we work with three metric spaces, namely  $X$ , its cone-off  $\dot{X}$  and the quotient  $\bar{X}$ . Since the map  $X \hookrightarrow \dot{X}$  is an embedding we use the same letter  $x$  to designate a point of  $X$  and its image in  $\dot{X}$ . We write  $\bar{x}$  for its image in  $\bar{X}$ . Unless stated otherwise, we keep the notation  $|\cdot|$  (without mentioning the space) for the distances in  $X$  or  $\bar{X}$ . The metric on  $\dot{X}$  will be denoted by  $|\cdot|_{\dot{X}}$ .

**The space  $\dot{X}$ .** Let us now fix  $\rho > 0$ . Its value will be made precise later (see Theorem 6.11). It should be thought as a very large parameter. We are going to build a cone-off over  $X$ . To measure the distortion between  $X$  and the spaces that we attach we use the map  $\mu : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  studied in Proposition 4.4. It is an  $a$ -comparison map with  $a = 1/24(1 + 1/\text{sh } \rho)$ . Let  $(H, Y) \in \mathcal{Q}$ . We denote by  $|\cdot|_Y$  the length metric on  $Y$  induced by the restriction of  $|\cdot|_X$  on  $Y$ . Since  $Y$  is strongly quasi-convex,  $Y$  endowed with  $|\cdot|_Y$  is a length space such that for all  $y, y' \in Y$

$$|y - y'|_X \leq |y - y'|_Y \leq |y - y'|_X + 8\delta. \quad (16)$$

We consider the cone  $Z(Y)$  of radius  $\rho$  over  $(Y, |\cdot|_Y)$ . It comes with a map  $\iota : Y \hookrightarrow Z(Y)$  as defined in Section 4. It follows from (16) and the properties of  $\mu$  that for every  $y, y' \in Y$ .

$$\mu(|y - y'|_X) \leq |\iota(y) - \iota(y')|_{Z(Y)} \leq \mu(|y - y'|_X) + 8\delta \quad (17)$$

The set  $\mathcal{Z}$  of all triples  $(Z(Y), Y, \iota)$  constructed in this way is a  $(\mu, X)$ -family.

**Definition 6.1.** The *cone-off* space  $\dot{X}(\mathcal{Z})$  or simply  $\dot{X}$  is the space obtained by attaching on  $X$  for every  $(Z, Y, \iota) \in \mathcal{Z}$  the cones  $Z(Y)$  along  $Y$  according to  $\iota$ .

By Proposition 5.10  $\dot{X}$  is a length space. Note that  $X$  is a deformation retract of  $\dot{X} \setminus v(\mathcal{Z})$ . Here  $v(\mathcal{Z})$  stands for the set of all apices of the cones  $Z(Y)$ .

**Lemma 6.2.** *The space  $\dot{X}$  is  $50\delta$ -simply-connected.*

*Proof.* Since  $X$  is a  $\delta$ -hyperbolic length space, it is  $50\delta$ -simply connected [6, Chap. 5, Prop. 1.1]. Moreover, the length of a loop contained in  $X$  is shorter measured with  $|\cdot|_{\dot{X}}$  than with  $|\cdot|_X$ . Therefore it is sufficient to show that any loop of  $\dot{X}$  is homotopic to a loop in  $X$ . Recall that the subsets  $Y$ 's over which we built the cones are path connected. Therefore any loop of  $\dot{X}$  can be homotoped to a loop avoiding the set of apices  $v(\mathcal{Z})$ . The conclusion follows from the fact that  $X$  is a deformation retract of  $\dot{X} \setminus v(\mathcal{Z})$ .  $\square$

The action of  $G$  on  $X$  extends by homogeneity in an action on  $\dot{X}$ : if  $x = (y, r)$  is a point of a cone  $Z(Y)$  and  $g$  an element of  $G$  then  $gx$  is the point of the cone  $Z(gY)$  defined by  $(gy, r)$ . It follows from the definition of  $|\cdot|_{\dot{X}}$  that  $G$  acts by isometries on  $\dot{X}$ .

**Lemma 6.3.** *Let  $x$  be a point of  $\dot{X}$  in the  $\alpha$ -neighborhood of  $X$ . The set  $S$  of elements  $g \in G$  such that  $|gx - x|_{\dot{X}} < 2(\rho - \alpha)$  is finite.*

*Proof.* We denote by  $p$  a projection of  $x$  on  $X$ . Let  $g \in S$ . The point  $gp$  is a projection of  $gx$  on  $X$ . By the triangle inequality

$$\mu(|gp - p|) \leq |gp - p|_{\dot{X}} \leq |gx - x|_{\dot{X}} + 2\alpha < 2\rho.$$

Thus  $|gp - p| < \pi \operatorname{sh} \rho$ . However the action of  $G$  on  $X$  is proper. Therefore the set of elements  $g \in G$  such that  $|gp - p| < \pi \operatorname{sh} \rho$  is finite, hence so is  $S$ .  $\square$

**Proposition 6.4.** *There exist universal positive numbers  $\rho_0 > 10^{20}\delta$ ,  $\delta_0$  and  $\Delta_0$  (i.e. which do not depend on  $X$ ,  $G$  or  $\mathcal{Q}$ ) with the following property. If  $\rho \geq \rho_0$ ,  $\delta \leq \delta_0$  and  $\Delta(\mathcal{Q}) \leq \Delta_0$  then  $\dot{X}$  is  $900\delta$ -hyperbolic.*

**Remark.** Recall that  $\delta$  is the hyperbolicity constant of the hyperbolic plane  $\mathbf{H}_2$ .

*Proof.* The proof falls in two steps. First we show that  $\dot{X}$  is locally hyperbolic and then we apply the Cartan-Hadamard theorem. We first fix  $\rho_0 > 0$  such that  $\rho_0 > 10^{20}\delta$  and  $(1 + 1/\operatorname{sh} \rho_0)/24 \leq 1/50$ . Since  $\rho \geq \rho_0$  the map  $\mu$  is a  $1/50$ -comparison map. On the other hand we know that for every  $t \in [0, \pi \operatorname{sh} \rho]$ ,  $\mu(t) \geq 2 \operatorname{argsh}(t/\pi)$  (see Proposition 4.4(ii)). In particular there exists  $t > 0$  which only depends on  $\rho_0$  and  $\delta$  such that  $\mu(t) > 8 \cdot 10^{19}\delta$ . We denote by  $\delta_0(2\delta, \delta, 1/50, t)$  and  $\Delta_0(2\delta, \delta, 1/50, t)$  the constants provided by Theorem 5.27. All the cones that we attach are  $2\delta$ -hyperbolic length spaces. Moreover, the distortion is control by the inequality (17). Assume now that

$$\begin{aligned} \delta &\leq \min \left\{ \delta, \delta_0(2\delta, \delta, 1/50, t) \right\}, \\ \Delta(\mathcal{Q}) &\leq \Delta_0(2\delta, \delta, 1/50, t). \end{aligned}$$

Then by Theorem 5.27, every ball of radius  $\mu(t)/8$  of  $\dot{X}$  is  $3\delta$ -hyperbolic. By Lemma 6.2,  $\dot{X}$  is also  $50\delta$ -simply-connected. However  $\mu(t)/8 > 10^{19}\delta$ . It follows from the Cartan-Hadamard Theorem (Theorem A.1) that  $\dot{X}$  is (globally)  $900\delta$ -hyperbolic.  $\square$

**A rotation family.** To every  $(H, Y) \in \mathcal{Q}$  we associate the pair  $(H, v)$  where  $v$  is the apex of the cone  $Z(Y)$  viewed as a point of  $\dot{X}$ . We denote by  $\mathcal{R}$  the set of all pairs obtained in this way. Using the notation of Section 3 the set of apices  $v(\mathcal{R})$  is the same as  $v(\mathcal{Z})$ .

**Lemma 6.5.** *If  $T(\mathcal{Q}) \geq \pi \operatorname{sh} \rho$  then the collection  $\mathcal{R}$  is a  $\sigma$ -rotation family where  $\sigma = 2\rho$ .*

*Proof.* Let  $(H, v)$  be a pair of  $\mathcal{R}$ . It corresponds to a pair  $(H, Y)$  of  $\mathcal{Q}$  where  $v$  is the apex of the cone  $Z(Y)$ . We assumed that  $H$  stabilizes  $Y$ , thus it fixes  $v$ . By Lemma 5.7 the metric of  $Z(Y)$  and  $\dot{X}$  coincide on the ball  $B(v, \rho/5)$ . Since  $T(\mathcal{Q}) \geq \pi \operatorname{sh} \rho$  Lemma 4.7 implies that for every  $x \in B(v, \rho/5)$ , for every  $h \in H \setminus \{1\}$   $|hx - x|_{\dot{X}} = 2|x - v|_{\dot{X}}$ . This proves Axiom (R1) of the definition of rotation family (see Definition 3.1) Axiom (R2) follows from the fact that the distance between two apices in  $\dot{X}$  is at least  $2\rho$ . Finally the family  $\mathcal{Q}$  being  $G$ -invariant, so is  $\mathcal{R}$ .  $\square$

**The space  $\bar{X}$ .** Recall that  $K$  is the (normal) subgroup of  $G$  generated by the subgroups  $H$  with  $(H, Y) \in \mathcal{Q}$ .

**Definition 6.6.** The space  $\bar{X}$  is the quotient of the cone off  $\dot{X}$  by the subgroup  $K$ .

**Proposition 6.7.** *There exist universal positive numbers  $\rho_0 > 10^{20}\delta$ ,  $\delta_0$  and  $\Delta_0$  (i.e. which do not depend on  $X$ ,  $G$  or  $\mathcal{Q}$ ) with the following property. Assume that  $\rho \geq \rho_0$  and  $\delta \leq \delta_0$ . If in addition  $\Delta(\mathcal{Q}) \leq \Delta_0$  and  $T(\mathcal{Q}) \geq \pi \operatorname{sh} \rho$  then  $\bar{X}$  is a  $\bar{\delta}$ -hyperbolic space where  $\bar{\delta} \leq 64 \cdot 10^4 \delta$ . The group  $\bar{G}$  acts by isometries on it and for all  $(H, v) \in \mathcal{R}$  the projection  $G \rightarrow \bar{G}$  induces an isomorphism from  $\operatorname{Stab}(v)/H$  onto  $\operatorname{Stab}(\bar{v})$ .*

*Proof.* The constant  $\sigma_0 = \sigma_0(900\delta)$  is the one given by the fundamental theorem of rotation families (Theorem 3.2). We denote by  $\delta_0$ ,  $\Delta_0$  and  $\rho_0$  the parameters given by Proposition 6.4. By increasing if necessary  $\rho_0$  we may assume that  $2\rho_0 \geq \sigma_0$ . It follows from Proposition 6.4 that  $\dot{X}$  is  $900\delta$ -hyperbolic. According to Lemma 6.5 the collection  $\mathcal{R}$  that we previously built is a  $2\rho$ -rotation family. However by assumption  $2\rho \geq \sigma_0$ , thus we can apply all the results from Section 3 about rotation families. Among others the space  $\bar{X}$  is  $\bar{\delta}$ -hyperbolic with  $\bar{\delta} \leq 64 \cdot 10^4 \delta$ . Moreover,  $\bar{G}$  acts by isometries on it. The last statement is a consequence of Corollary 3.13.  $\square$

For the remainder of this paragraph we assume that  $X$ ,  $G$  and  $\mathcal{Q}$  satisfy the assumptions of Proposition 6.7.

**Lemma 6.8.** *Let  $v \in v(\mathcal{R})$ . Let  $\bar{g} \in \bar{G} \setminus \operatorname{Stab}(\bar{v})$ . For every  $x \in \bar{X}$ ,  $|\bar{g}\bar{x} - \bar{x}| \geq 2(\rho - |\bar{x} - \bar{v}|)$ .*

*Proof.* Since  $\bar{g}$  does not fix  $\bar{v}$  the distance between  $\bar{v}$  and  $\bar{g}\bar{v}$  is at least  $2\rho$ . It follows the triangle inequality that

$$2\rho \leq |\bar{g}\bar{v} - \bar{g}\bar{x}| + |\bar{g}\bar{x} - \bar{x}| + |\bar{x} - \bar{v}| = |\bar{g}\bar{x} - \bar{x}| + 2|\bar{x} - \bar{v}|. \quad \square$$

**Proposition 6.9.** *The group  $\bar{G}$  acts properly co-compactly on  $\bar{X}$ .*

*Proof.* Let  $\bar{x}$  be a point of  $\bar{X}$ . We claim that the set of elements  $\bar{g} \in \bar{G}$  such that  $\bar{g}B(\bar{x}, \bar{\delta}) \cap B(\bar{x}, \bar{\delta}) \neq \emptyset$  is finite. This will prove that the action is proper. We distinguish two cases.

**Case 1.** *There exists  $(H, v) \in \mathcal{R}$  such that  $|\bar{v} - \bar{x}| < \rho - \bar{\delta}$ . Let  $\bar{g} \in \bar{G}$ . Assume that  $B(\bar{x}, \bar{\delta})$  intersects  $\bar{g}B(\bar{x}, \bar{\delta})$ . In particular  $|\bar{g}\bar{x} - \bar{x}| \leq 2\bar{\delta}$ . According to Lemma 6.8,  $\bar{g}$  fixes  $\bar{v}$ . However  $\operatorname{Stab}(\bar{v})$  is isomorphic to  $\operatorname{Stab}(v)/H = \operatorname{Stab}(Y)/H$  which is finite ( $H$  acts co-compactly on  $Y$ ). Consequently,  $\bar{g}$  belongs to  $\operatorname{Stab}(\bar{v})$ , which is finite.*



**Case 2.** *The point  $\bar{x}$  is  $\bar{\delta}$ -close to  $\nu(X)$ .* We denote by  $x$  a pre-image of  $\bar{x}$  in  $\dot{X}$  and by  $S$  the set of elements  $g \in G$  such that  $|gx - x|_{\dot{X}} \leq 2\bar{\delta}$ . Recall that  $\rho \geq \rho_0 > 2\bar{\delta}$ . According to Lemma 6.3,  $S$  is finite. Let  $\bar{g} \in \bar{G}$  such that  $\bar{g}B(\bar{x}, \bar{\delta})$  intersects  $B(\bar{x}, \bar{\delta})$ . In particular  $|\bar{g}\bar{x} - \bar{x}| < 2\bar{\delta}$ . By Proposition 3.15 there exists  $g \in G$  such that  $|gx - x|_{\dot{X}} = |\bar{g}\bar{x} - \bar{x}|$ . Thus  $\bar{g}$  belongs to the image in  $\bar{G}$  of  $S$ , which is finite.

The space  $\bar{X}/\bar{G}$  can be obtained by attaching on  $X/G$  finitely many cones of radius  $\rho$  over  $Y/H$  where  $(H, Y) \in \mathcal{Q}/G$ . As  $G$  (respectively  $H$ ) acts co-compactly on  $X$  (respectively  $Y$ ) the space  $X/G$  (respectively  $Y/H$ ) is compact. Consequently, so is  $\bar{X}/\bar{G}$ . Thus the action of  $\bar{G}$  on  $\bar{X}$  is co-compact.  $\square$

**Proposition 6.10.** *The space  $\bar{X}$  is proper and geodesic*

*Proof.* The space  $\bar{X}$  is a metric space endowed with an action of  $\bar{G}$  which is proper and co-compact. It follows that  $\bar{X}$  is complete and locally compact [4, Chap. I.8, Ex. 8.4(1)]. On the other hand  $\dot{X}$  and thus  $\bar{X}$  is a length space. The Hopf-Rinow Theorem implies that  $\bar{X}$  is geodesic [4, Chap.1, Prop. 3.7].  $\square$

**Small cancellation theorem** The previous results can be summarized in the following theorem. It is an analog of the well-known fact saying that a group whose presentation satisfies the usual  $C''(\lambda)$  small cancellation assumption with  $\lambda < 1/6$  is hyperbolic (see [11, Appendix, Th.36]).

**Theorem 6.11** (Small cancellation theorem). *There exist positive constants  $\rho_0$ ,  $\delta_0$  and  $\Delta_0$  satisfying the following property. Let  $G$  be a group acting properly co-compactly on a geodesic proper  $\delta$ -hyperbolic space. Let  $\mathcal{Q}$  be a family of pairs  $(H, Y)$  such that  $Y$  is a strongly quasi-convex subset of  $X$  and  $H$  a subgroup of  $G$  stabilizing  $Y$ . We assume that*

- (i) *for every  $(H, Y) \in \mathcal{Q}$ ,  $H$  acts co-compactly on  $Y$ ,*
- (ii)  *$\mathcal{Q}$  is  $G$ -invariant i.e., for every  $(H, Y) \in \mathcal{Q}$  for every  $g \in G$   $(gHg^{-1}, gY)$  is still an element of  $\mathcal{Q}$ ,*
- (iii) *the quotient  $\mathcal{Q}/G$  is finite.*

*Let  $K$  be the (normal) subgroup of  $G$  generated by the subgroups  $H$ 's where  $(H, Y) \in \mathcal{Q}$ . Let  $\rho \geq \rho_0$ . Let  $\dot{X}$  be the cone-off space obtained by attaching for every  $(H, Y) \in \mathcal{Q}$  a cone of radius  $\rho$  and base  $Y$  on  $X$ . Let  $\bar{X}$  be the quotient of  $\dot{X}$  by  $K$ .*

*If  $\delta \leq \delta_0$ ,  $\Delta(\mathcal{Q}) \leq \Delta_0$  and  $T(\mathcal{Q}) \geq \pi \operatorname{sh} \rho$  then  $\bar{X}$  is a geodesic proper  $\bar{\delta}$ -hyperbolic space with  $\bar{\delta} \leq 64 \cdot 10^4 \delta$ . The group  $\bar{G} = G/K$  acts properly co-compactly by isometries on  $\bar{X}$ . For every  $(H, Y) \in \mathcal{Q}$ , the projection  $G \rightarrow \bar{G}$  induces an embedding  $\operatorname{Stab}(Y)/H \hookrightarrow \bar{G}$ .*

**Remarks.** Note that in the theorem the constants  $\delta_0$  and  $\Delta_0$  (respectively  $\rho_0$ ) can be chosen arbitrary small (respectively large). From now on, we will always assume that  $\rho_0 > 10^{20}\delta$  whereas  $\delta_0, \Delta_0 < 10^{-10}\delta$ . These estimates are absolutely not optimal. We chose them very generously to be sure that all the inequalities that we might need later will be satisfied. What really matters is their orders of magnitude recalled below.

$$\max \{\delta_0, \Delta_0\} \ll \delta \ll \rho_0 \ll \pi \operatorname{sh} \rho_0.$$

An other important point to remember is the following. The constants  $\delta_0$ ,  $\Delta_0$  and  $\pi \operatorname{sh} \rho_0$  are used to describe the geometry of  $X$  whereas  $\delta$  and  $\rho_0$  refers to the one of  $\dot{X}$  or  $\bar{X}$ . This theorem looks slightly different from Theorem 1.4 given in the introduction. The later follows from a rescaling argument. However we prefer to retain that formulation which will make easier to keep track of the parameters. In Section 7.2 we give an example where we apply this rescaling argument.

**Comparison with the original proof.** At this stage, we can explain precisely the main differences between our proof and the one by T. Delzant and M. Gromov. In [10], the authors construct an orbifold  $\bar{Q}$  by attaching on  $X/G$  appropriate cones. In this context the space  $\dot{X}$  can be used as a chart for the neighborhood of the points of  $\bar{Q}$  close to  $X/G$ . They interpret  $\bar{G}$  as the fundamental group (in the sense of orbifolds) of  $\bar{Q}$ . In this description the groups  $\operatorname{Stab}(Y)/H$  correspond to the stabilizers of the singularities of the orbifold. They use then a version of the Cartan-Hadamard theorem for orbifolds ([10, Th. 4.3.1]). They prove that if an orbifold is locally hyperbolic (with the correct quantifiers) then it is developable and its fundamental cover  $\bar{X}$  (in the sense of orbifolds) is globally hyperbolic. One consequence of the developability is that the stabilizers of the singularities, namely  $\operatorname{Stab}(Y)/H$ , embeds into  $\bar{G}$ . An other is that the developing map  $\nu : \dot{X} \rightarrow \bar{X}$  induces a local isometry. In our approach we substitute the topological point of view for a study of rotation families acting on  $\dot{X}$ . It allows us to construct directly  $\bar{X}$  as a quotient of  $\dot{X}$  without going through  $\bar{Q}$ . The two consequences of developability mentioned before correspond to our Corollary 3.13 and Proposition 3.15. T. Delzant and M. Gromov also used the description in terms of orbifold to lift elements from  $\bar{G}$  to  $G$ . In particular if two paths of  $\bar{X}$  with the same extremities are homotopic relative to their endpoints in  $\bar{X} \setminus \bar{v}(\mathcal{R})$ , then their lifts in  $\dot{X}$  will define the same endpoint in  $\dot{X}$ . We prove instead Propositions 3.20 and 3.21. An other difference concerns the framework used to describe the geometry of  $\dot{X}$  and  $\bar{X}$ . We only proved that these spaces were hyperbolic. They are actually endowed with a finer metric structure. As explained in [10] they are locally  $\operatorname{CAT}(-1, \varepsilon)$ , which roughly means that every ball of large radius satisfies the  $\operatorname{CAT}(-1)$  condition up to an error  $\varepsilon$ . However this refinement is not needed to prove the infiniteness of Burnside groups.

For the remainder of Section 6 we assume that  $X$ ,  $G$  and  $\mathcal{Q}$  are as in Theorem 6.11.

## 6.2 Isometries of the quotient

Since  $\bar{G}$  acts properly co-compactly by isometries on a hyperbolic space, its elements are either elliptic or hyperbolic. In this section we study how the type of an isometry is related to the one of its preimage in  $G$  (seen as an isometry of  $X$ ).

**Proposition 6.12.** *Let  $\bar{F}$  be a finite subgroup of  $\bar{G}$ . Either there exists  $v \in v(\mathcal{R})$  such that  $\bar{F}$  is contained in  $\operatorname{Stab}(\bar{v})$  or  $\bar{F}$  is isomorphic to a finite subgroup of  $G$ .*

*Proof.* Recall that  $C_{\bar{F}}$  is the set of points  $\bar{x} \in \bar{X}$  such that for every  $\bar{g} \in \bar{F}$ ,  $|\bar{g}\bar{x} - \bar{x}| \leq 10\bar{\delta}$  (see Definition 2.30). It is  $\bar{F}$ -invariant and  $8\bar{\delta}$ -quasi-convex (see Corollary 2.32). If  $\bar{F}$  is not contained in some  $\operatorname{Stab}(\bar{v})$ , then according to Lemma 6.8 for every  $v \in v(\mathcal{R})$ ,  $C_{\bar{F}}$  does not intersect  $B(\bar{v}, \rho - 5\bar{\delta})$ . By Proposition 3.21, there exists a subset  $C$  of  $\dot{X}$  such that the map  $\nu : \dot{X} \rightarrow \bar{X}$  induces an isometry from  $C$  onto  $C_{\bar{F}}$  and the projection  $\pi : G \rightarrow \bar{G}$  induces an isomorphism from  $\operatorname{Stab}(C)$  onto  $\operatorname{Stab}(C_{\bar{F}})$ . In particular  $\bar{F}$  is isomorphic to a subgroup  $F$  of  $\operatorname{Stab}(C)$ .  $\square$

**Proposition 6.13.** *We denote by  $l$  the greatest lower bound on the stable translation length (in  $X$ ) of hyperbolic elements of  $G$  which do not belong to some  $\text{Stab}(v)$ ,  $v \in v(\mathcal{R})$ . Then  $r_{\text{inj}}(\bar{G}, \bar{X}) \geq \min\{\kappa l/8, \bar{\delta}\}$  where  $\kappa = 2\rho/\pi \text{sh } \rho$ .*

*Proof.* Let  $\bar{g}$  be a hyperbolic element of  $\bar{G}$ . Recall that for every  $m \in \mathbf{N}$  we have  $m[\bar{g}]^\infty \geq [\bar{g}^m] - 32\bar{\delta}$ . Therefore it suffices to find an integer  $m$  such that  $[\bar{g}^m] \geq m \min\{\kappa l/8, \bar{\delta}\} + 32\bar{\delta}$ . We denote by  $m$  the largest integer satisfying  $m \min\{\kappa l/8, \bar{\delta}\} \leq 40\bar{\delta}$ . Assume that  $[\bar{g}^m]$  is smaller than  $m \min\{\kappa l/8, \bar{\delta}\} + 32\bar{\delta}$ . In particular  $[\bar{g}^m] \leq 72\bar{\delta}$ . However  $\bar{g}$  is hyperbolic thus it cannot belong to some  $\text{Stab}(\bar{v})$ . According to Lemma 6.8 for every  $v \in v(\mathcal{R})$ ,  $A_{\bar{g}^m}$  does not intersect  $B(\bar{v}, \rho - 36\bar{\delta})$ . By Proposition 3.21, there exists a subset  $A$  of  $\dot{X}$  such that the map  $\nu : \dot{X} \rightarrow \bar{X}$  induces an isometry from  $A$  onto  $A_{\bar{g}^m}$  and the projection  $\pi : G \rightarrow \bar{G}$  induces an isomorphism from  $\text{Stab}(A)$  onto  $\text{Stab}(A_{\bar{g}^m})$ . We denote by  $g$  the preimage of  $\bar{g}$  in  $\text{Stab}(A)$ . It is a hyperbolic element of  $G$ . In particular  $[g]^\infty \geq l$ . Let  $\bar{x}$  be a point of  $A_{\bar{g}^m}$ ,  $x$  the preimage of  $\bar{x}$  in  $A$  and  $y$  a projection of  $x$  on  $X$ . In particular  $g^m y$  is a projection of  $g^m x$  on  $X$  and  $|x - y|_{\dot{X}} \leq 36\bar{\delta}$ . Moreover, we have

$$\mu(|g^m y - y|) \leq |g^m x - x|_{\dot{X}} + 72\bar{\delta} = |\bar{g}^m \bar{x} - \bar{x}| + 72\bar{\delta} \leq \max\{[\bar{g}^m], 8\bar{\delta}\} + 72\bar{\delta} \leq 144\bar{\delta} < 2\rho.$$

It follows that  $|g^m y - y| < \pi \text{sh } \rho$ . The function  $\mu$  being concave we get  $ml \leq |g^m y - y| \leq 144\kappa^{-1}\bar{\delta}$ , which contradicts the maximality of  $m$ .  $\square$

### 6.3 Groups without even-torsion

This section is specific to the study of Burnside groups of odd exponents. As explained in the introduction, we will construct by induction a sequence of groups by adjoining at each steps relations of the form  $r^n$ . If  $n$  is large enough, these relations satisfy a small cancellation assumption. The difficulty is to prove that the same exponent  $n$  can be used at every step. This can be achieved by estimating the invariant  $A$ . More precisely we provide in this section an upper bound for  $A(\bar{G}, \bar{X})$  in terms of  $A(G, X)$  and  $\bar{\delta}$ . To that end we make the following hypotheses.

- (i) Every elementary subgroup of  $G$  is cyclic - infinite or of finite of odd order.
- (ii) For every  $(H, Y) \in \mathcal{Q}$ , there exists  $r \in G$  which is hyperbolic and not a proper power such that  $Y$  is the cylinder of  $r$  (see definition in Section 2.5) and  $H$  the cyclic subgroup of  $G$  generated by an odd power of  $r$ .

We still require that  $X$ ,  $G$  and  $\mathcal{Q}$  satisfy the assumptions of Theorem 6.11.

**Proposition 6.14.** *Let  $\bar{g}$  and  $\bar{h}$  be two elements of  $\bar{G}$  such that  $[\bar{g}]$  and  $[\bar{h}]$  are at most  $1000\bar{\delta}$ . One of the following holds.*

- (i) *There exists  $v \in v(\mathcal{R})$  such that  $\bar{g}$  and  $\bar{h}$  belong to  $\text{Stab}(\bar{v})$ .*
- (ii) *There exist respective preimages  $g$  and  $h$  in  $G$  of  $\bar{g}$  and  $\bar{h}$  such that  $[g]$  and  $[h]$  are at most  $\pi \text{sh}(1034\bar{\delta})$  and*

$$\text{diam}\left(A_g^{+17\bar{\delta}} \cap A_h^{+17\bar{\delta}}\right) \leq \text{diam}\left(A_g^{+17\bar{\delta}} \cap A_h^{+17\bar{\delta}}\right) + \pi \text{sh}(2087\bar{\delta}).$$

**Remarks.** In the statement of the proposition all the metric objects are measured either with the distance of  $X$  or  $\bar{X}$ , but not with the one of  $\dot{X}$ . Note that this result actually holds without our additional assumptions on the torsion.

*Proof.* Without loss of generality we can assume that the intersection of the respective  $17\bar{\delta}$ -neighborhoods of  $A_{\bar{g}}$  and  $A_{\bar{h}}$  is not empty. Let us call  $\bar{Z}$  this intersection. Assume that there exists  $v \in v(\mathcal{R})$  and a point  $\bar{z} \in \bar{Z}$  such that  $|\bar{v} - \bar{z}| < \rho - 517\bar{\delta}$ . By definition  $\bar{g}$  and  $\bar{h}$  move  $\bar{z}$  by a distance at most  $1034\bar{\delta}$ . According to Lemma 6.8, they belong to  $\text{Stab}(\bar{v})$ , which provides the first case.

We now assume that for every  $v \in v(\mathcal{R})$ ,  $\bar{Z}$  does not intersect  $B(\bar{v}, \rho - 517\bar{\delta})$ . By Lemma 2.12,  $\bar{Z}$  is  $7\bar{\delta}$ -quasi-convex. Moreover,  $\bar{g}$  and  $\bar{h}$  move any point of  $\bar{Z}$  by at most  $1034\bar{\delta}$ . According to Proposition 3.21, there exists a subset  $Z$  of  $\dot{X}$  and respective preimages  $g$  and  $h$  of  $\bar{g}$  and  $\bar{h}$  satisfying the following properties.

- (i) The map  $\nu : \dot{X} \rightarrow \bar{X}$  induces an isometry from  $Z$  onto  $\bar{Z}$ .
- (ii) For every  $z \in Z$  we have  $|gz - z|_{\dot{X}} = |\bar{g}\bar{z} - \bar{z}|$  and  $|hz - z|_{\dot{X}} = |\bar{h}\bar{z} - \bar{z}|$ .

We now denote by  $\bar{z}$  and  $\bar{z}'$  two points of  $\bar{Z}$  such that

$$|\bar{z} - \bar{z}'| \geq \text{diam} \left( A_{\bar{g}}^{+17\bar{\delta}} \cap A_{\bar{h}}^{+17\bar{\delta}} \right) - \bar{\delta}.$$

The points  $z$  and  $z'$  stands for their preimages in  $Z$ . We write  $x$  and  $x'$  for respective projections of  $z$  and  $z'$  on  $X$ . By assumption,  $\bar{Z}$  lies in the  $517\bar{\delta}$ -neighborhood of  $\nu(X)$ . Thus  $|x - z|_{\dot{X}}, |x' - z'|_{\dot{X}} \leq 517\bar{\delta}$ . In particular,

$$\mu(|gx - x|) \leq |gx - x|_{\dot{X}} \leq |\bar{g}\bar{z} - \bar{z}| + 1034\bar{\delta} \leq 2068\bar{\delta} < \pi \text{sh} \rho.$$

It follows that  $|gx - x| \leq \pi \text{sh}(1034\bar{\delta})$ . The same holds for  $x'$ . Consequently,  $[g] \leq \pi \text{sh}(1034\bar{\delta})$ . Moreover,  $x$  and  $x'$  belong to the  $A$ -neighborhood of  $A_{\bar{g}}$  with  $A = \pi \text{sh}(1034\bar{\delta})/2 + 7\bar{\delta}$ . Similarly they belong to the  $A$ -neighborhood of  $A_{\bar{h}}$ . By Lemma 2.13

$$|x - x'| \leq \text{diam} \left( A_{\bar{g}}^{+17\bar{\delta}} \cap A_{\bar{h}}^{+17\bar{\delta}} \right) + \pi \text{sh}(1034\bar{\delta}) + 14\bar{\delta} + 4\bar{\delta}$$

On the other hand, the map  $X \rightarrow \dot{X}$  shorten the distances. Therefore

$$|x - x'| \geq |x - x'|_{\dot{X}} \geq |z - z'|_{\dot{X}} - 1034\bar{\delta} \geq |\bar{z} - \bar{z}'| - 1034\bar{\delta} \geq \text{diam} \left( A_{\bar{g}}^{+17\bar{\delta}} \cap A_{\bar{h}}^{+17\bar{\delta}} \right) - 1035\bar{\delta}.$$

The conclusion of the second case follows from the last two inequalities.  $\square$

**Corollary 6.15.** *The invariant  $A(\bar{G}, \bar{X})$  is bounded above by  $A(G, X) + \pi \text{sh}(10^4\bar{\delta})$ .*

*Proof.* We denote by  $\mathcal{A}$  the set of pairs  $(\bar{g}, \bar{h})$  of  $\bar{G}$  generating a non-elementary subgroup such that  $[\bar{g}]$  and  $[\bar{h}]$  are at most  $1000\bar{\delta}$ . Let  $(\bar{g}, \bar{h}) \in \mathcal{A}$ . By definition of  $\mathcal{A}$ ,  $\bar{g}$  and  $\bar{h}$  cannot be both in some  $\text{Stab}(\bar{v})$ . According to Proposition 6.14, there exist respective preimages  $g$  and  $h$  in  $G$  of  $\bar{g}$  and  $\bar{h}$  such that  $[g]$  and  $[h]$  are at most  $\pi \text{sh}(1034\bar{\delta})$  and

$$\text{diam} \left( A_{\bar{g}}^{+17\bar{\delta}} \cap A_{\bar{h}}^{+17\bar{\delta}} \right) \leq \text{diam} \left( A_g^{+17\bar{\delta}} \cap A_h^{+17\bar{\delta}} \right) + \pi \text{sh}(2087\bar{\delta}).$$

However we assumed that every elementary subgroup of  $G$  is cyclic. Thus by Proposition 2.41,

$$\text{diam} \left( A_{\bar{g}}^{+17\bar{\delta}} \cap A_{\bar{h}}^{+17\bar{\delta}} \right) \leq 2([g] + [h]) + A(G, X) + 667\bar{\delta} \leq A(G, X) + \pi \text{sh}(10^4 \bar{\delta}).$$

The last inequality holds for every  $(\bar{g}, \bar{h}) \in \mathcal{A}$ , which leads to the result.  $\square$

**Lemma 6.16.** *If  $T(\mathcal{Q}) \geq 4\pi \text{sh } \rho$  then an element  $\bar{g} \in \bar{G} \setminus \{1\}$  cannot fix more than one apex of  $\bar{v}(\mathcal{R})$ .*

*Proof.* Assume that  $\bar{g}$  fixes an apex of  $\bar{X}$ . There exists  $(H, Y) \in \mathcal{Q}$  such that the apex  $\bar{v}$  fixed by  $\bar{g}$  is the image in  $\bar{X}$  of the apex of the cone  $Z(Y)$ . We proved that  $\text{Stab}(\bar{v})$  is isomorphic to  $\text{Stab}(v)/H = \text{Stab}(Y)/H$ . However  $\text{Stab}(Y)$  is a cyclic group generated by  $r \in G$ . Therefore there exists  $m \in \mathbf{Z}$  such that  $\bar{g}$  is the image of  $g = r^m$ . Since  $\bar{g}$  is not trivial  $m \not\equiv 0 \pmod{n}$ . Thus there exists an integer  $p \in m\mathbf{Z} + n\mathbf{Z}$  between  $n/3$  and  $2n/3$ . In particular  $\bar{r}^p$  is a power of  $\bar{g}$ . Let  $x = (y, r)$  be a point of the cone  $Z(Y)$ . By construction of  $p$  we have

$$|r^p y - y| \geq p[r]^\infty \geq n[r]^\infty/3 \geq T(\mathcal{Q})/3 - 32\bar{\delta} \geq \pi \text{sh } \rho.$$

Consequently,  $|r^p x - x|_{\bar{X}} = 2r$ . On the other hand  $y$  is a point of the cylinder of  $r$  and thus is contained in the  $36\bar{\delta}$ -neighborhood of the axis of  $r^p$  (see Lemma 2.27). Hence

$$|r^p y - y| \leq [r^p] + 72\bar{\delta} \leq p[r]^\infty + 104\bar{\delta} \leq [r^n] + 104\bar{\delta} - (n - p)[r]^\infty \leq [r^n] - \pi \text{sh } \rho.$$

According to Lemma 4.8,  $|\bar{r}^p \bar{x} - \bar{x}| = |r^p x - x|_{\bar{X}} = 2r$ . It follows that the points of  $Z(Y)/H$  contained in the axis of  $\bar{r}^p$  are  $4\bar{\delta}$ -close to  $\bar{v}$ . This axis is nevertheless  $14\bar{\delta}$ -quasi-convex. Thus it is contained in  $B(\bar{v}, 5\bar{\delta})$ . In particular,  $\bar{r}^p$  cannot fix an other apex than  $\bar{v}$ . Hence  $\bar{g}$  cannot either.  $\square$

**Lemma 6.17.** *The group  $\bar{G}$  has no element of order 2.*

*Proof.* According to Proposition 6.12, every element of  $\bar{G}$  with finite order is contained either in the image of a finite subgroup of  $G$  or in some  $\text{Stab}(\bar{v})$ , with  $v \in v(\mathcal{R})$ . It follows from our assumptions at the beginning of the section that none of these groups contain an element of order 2. Thus  $\bar{G}$  cannot have even-torsion.  $\square$

**Proposition 6.18.** *If  $T(\mathcal{Q}) \geq 4\pi \text{sh } \rho$  then every elementary subgroup of  $\bar{G}$  is cyclic.*

*Proof.* Let  $\bar{E}$  be an elementary subgroup of  $\bar{G}$ . Assume first that  $\bar{E}$  is finite. By Proposition 6.12  $\bar{E}$  is either isomorphic to a subgroup of  $G$  or contained in some  $\text{Stab}(\bar{v}) = \text{Stab}(v)/H$  with  $(H, v) \in \mathcal{R}$ . However we assumed that every elementary subgroup of  $G$  is cyclic. On the other hand if  $v$  is the apex of the cone built on  $Y$  with  $(H, Y) \in \mathcal{Q}$  then  $\text{Stab}(\bar{v})$  is isomorphic to  $\text{Stab}(Y)/H$ . According to our assumption  $\text{Stab}(Y)/H$  is also cyclic. Hence so is  $\bar{E}$ .

Suppose now that  $\bar{E}$  is infinite. By Lemma 6.17 it does not contain an element of order 2. Therefore  $\bar{E}$  is isomorphic to the semi-direct product  $\bar{F} \rtimes \mathbf{Z}$  where  $\bar{F}$  is the maximal finite subgroup of  $\bar{E}$ ,  $\mathbf{Z}$  is generated by a hyperbolic element  $\bar{g}$  acting by conjugation on  $\bar{F}$ . We claim that the cylinder  $Y_{\bar{g}}$  of  $\bar{g}$  lies in the  $53\bar{\delta}$ -neighborhood of  $\nu(X)$ . Assume on the contrary that there exists  $v \in v(\mathcal{R})$  and a point  $\bar{x} \in Y_{\bar{g}}$  such that  $|\bar{x} - \bar{v}| < \rho - 53\bar{\delta}$ . By Lemma 2.34  $Y_{\bar{g}}$  lies in the  $48\bar{\delta}$ -neighborhood of  $C_{\bar{F}}$ . Thus for every  $\bar{u} \in \bar{F}$ ,  $|\bar{u}\bar{x} - \bar{x}| \leq 106\bar{\delta}$ . According to Lemma 6.8  $\bar{F}$  is contained in  $\text{Stab}(\bar{v})$ . However  $\bar{g}$  normalizes  $\bar{F}$ , therefore  $\bar{F}$  also fixes  $\bar{g}\bar{v}$ . It follows from Lemma 6.16 that  $\bar{g}\bar{v} = \bar{v}$ . In

particular  $\bar{g}$  cannot be hyperbolic. Contradiction. The cylinder  $Y_{\bar{g}}$  is a  $2\bar{\delta}$ -quasi-convex subset of  $\bar{X}$  contained in the  $53\bar{\delta}$ -neighborhood of  $\nu(X)$ . Applying Proposition 3.21, there exists a subset  $Y$  of  $\bar{X}$  such that

- (i) The map  $\nu : \dot{X} \rightarrow \bar{X}$  induces an isometry from  $Y$  onto  $Y_{\bar{g}}$
- (ii) The projection  $G \rightarrow \bar{G}$  induces an automorphism from  $\text{Stab}(Y)$  onto  $\text{Stab}(Y_{\bar{g}})$ .

However  $\bar{E}$  stabilizes  $Y_{\bar{g}}$  thus it is isomorphic to a subgroup of  $G$ , which is necessarily elementary. All the elementary subgroups of  $G$  being cyclic, so is  $\bar{E}$ .  $\square$

**Proposition 6.19.** *Assume that there exists two isometries  $u, v \in G$  and a point  $x \in X$  such that  $0 < |uvu^{-1}v^{-1}x - x| < \rho/20$ . Then the same holds in  $\bar{X}$ . Moreover  $\bar{G}$  is not elementary.*

*Proof.* In order to simplify the notations we put  $g = uvu^{-1}v^{-1}$ . The natural map  $X \rightarrow \bar{X}$  shortens the distances. More precisely we have

$$0 < \mu(|gx - x|) \leq |gx - x|_{\bar{X}} \leq |gx - x| < \rho/20.$$

However the map  $\nu : \dot{X} \rightarrow \bar{X}$  induces an isometry from the ball of center  $x$  and radius  $\rho/20$  onto its image (see Proposition 3.15). Consequently,  $0 < |\bar{g}\bar{x} - \bar{x}| < \rho/20$ . Assume now that  $\bar{G}$  is elementary. By Proposition 6.18, it is cyclic and thus abelian. This contradicts the fact that  $|\bar{g}\bar{x} - \bar{x}| > 0$ .  $\square$

## 7 Applications

### 7.1 Periodic quotients of hyperbolic groups

The next proposition will play the role of the induction step in the proof of the main theorem.

**Proposition 7.1.** *There exist positive constants  $\rho_0, \delta_1$  and an integer  $n_0$  with the following properties. Let  $G$  be a group acting properly co-compactly by isometries on a geodesic proper  $\delta_1$ -hyperbolic space  $X$ . Let  $n_1 \geq n_0$  and  $n \geq n_1$  be an odd integer. We denote by  $R$  the set of hyperbolic elements  $r$  of  $G$  which are not a proper power and such that  $[r] \leq 1000\delta_1$ . Let  $K$  be the normal subgroup of  $G$  generated by  $\{r^n, r \in R\}$  and  $\bar{G}$  the quotient of  $G$  by  $K$ . We make the following assumptions.*

- (i)  *$G$  is non elementary. Every elementary subgroup of  $G$  is cyclic either infinite or finite with order dividing  $n$ .*
- (ii)  *$A(G, X) \leq 2\pi \text{sh}(10^4\delta_1)$ .*
- (iii)  *$r_{\text{inj}}(G, X) \geq 20\sqrt{\rho_0\delta_1/n_1}$ .*
- (iv) *There exist  $u, v \in G$  and a point  $x \in X$  such that*

$$0 < |uvu^{-1}v^{-1}x - x| \leq \rho_0/20.$$

*Then there exists a geodesic proper  $\delta_1$ -hyperbolic space  $\bar{X}$  on which  $\bar{G}$  acts properly, co-compactly by isometries. The action of  $\bar{G}$  on  $\bar{X}$  satisfies also Points (i)-(iv). Moreover, for every  $g \in G$ , if  $\bar{g}$  stands for its image in  $\bar{G}$  we have*

$$[\bar{g}]_{\bar{X}}^{\infty} \leq \frac{1}{\sqrt{n_1}} \left( \frac{\pi \text{sh } \rho_0}{5\sqrt{\rho_0\delta_1}} \right) [g]_X^{\infty}.$$

**Vocabulary.** Let  $G$  be a group acting by isometries on a space  $X$  and  $n$  an integer. Once  $n_1$  and  $n$  have been fixed, if  $G$  and  $X$  satisfy the assumption of the proposition including Points (i)-(iv), we will write that  $(G, X)$  satisfies the induction hypotheses for exponent  $n$ . The proposition says in particular that if  $(G, X)$  satisfies the induction hypotheses for exponent  $n$  then so does  $(\bar{G}, \bar{X})$ .

*Proof.* The parameters  $\rho_0$ ,  $\delta_0$  and  $\Delta_0$  are the one given by the small cancellation theorem (Theorem 6.11). We set  $\delta_1 = 64 \cdot 10^4 \delta$ . We now define the critical exponent  $n_0$ . To that end we consider a rescaling parameter  $\lambda_n$  depending on an integer  $n$

$$\lambda_n = \frac{\pi \operatorname{sh} \rho_0}{5\sqrt{n\rho_0\delta_1}}$$

The sequence  $(\lambda_n)$  converges to 0 as  $n$  approaches infinity. Therefore there exists an integer  $n_0$  such that for every  $n \geq n_0$

$$\lambda_n \delta_1 \leq \delta_0 \quad (18)$$

$$\lambda_n (2\pi \operatorname{sh}(10^4 \delta_1) + 86\delta_1) \leq \min \{ \Delta_0, \pi \operatorname{sh}(10^4 \delta_1) \} \quad (19)$$

$$\frac{100\lambda_n \rho_0 \delta_1}{\pi \operatorname{sh} \rho_0} \leq \delta_1 \quad (20)$$

$$\lambda_n \rho_0 \leq \rho_0 \quad (21)$$

Let  $n_1 \geq n_0$  and  $n \geq n_1$  be an odd integer. For simplicity of notation we denote by  $\lambda$  the rescaling parameter  $\lambda = \lambda_{n_1}$ . Let  $G$  be a group acting by isometries on a metric space  $X$  such that  $(G, X)$  satisfies the induction hypotheses for exponent  $n$ . We denote by  $R$  the set of hyperbolic elements  $r$  of  $G$  which are not a proper power and such that  $[r] \leq 1000\delta_1$ . Let  $K$  be the normal subgroup of  $G$  generated by  $\{r^n, r \in R\}$  and  $\bar{G}$  the quotient of  $G$  by  $K$ . We are going to prove that  $\bar{G}$  is a small cancellation quotient of  $G$ . To that end we consider the action of  $G$  on the rescaled space  $\lambda X$ . In particular it is a  $\delta$ -hyperbolic space, with  $\delta = \lambda\delta_1 \leq \delta_0$ . Unless stated otherwise, we will always work with the rescaled space  $\lambda X$ . We define the family  $\mathcal{Q}$  by

$$\mathcal{Q} = \left\{ (\langle r^n \rangle, Y_r) \mid r \in R \right\}.$$

**Lemma 7.2.** *The family  $\mathcal{Q}$  satisfies the following assumptions:  $\Delta(\mathcal{Q}) \leq \Delta_0$  and  $T(\mathcal{Q}) \geq 4\pi \operatorname{sh} \rho_0$ .*

*Proof.* We start with the upper bound of  $\Delta(\mathcal{Q})$ . Let  $r_1$  and  $r_2$  be two elements of  $R$  such that  $(\langle r_1^n \rangle, Y_{r_1}) \neq (\langle r_2^n \rangle, Y_{r_2})$ . According to Lemma 2.27,  $Y_{r_1}$  and  $Y_{r_2}$  are respectively contained in the  $36\delta$ -neighborhood of  $A_{r_1}$  and  $A_{r_2}$ , thus by Lemma 2.13

$$\operatorname{diam}(Y_{r_1}^{+5\delta} \cap Y_{r_2}^{+5\delta}) \leq \operatorname{diam}(A_{r_1}^{+17\delta} \cap A_{r_2}^{+17\delta}) + 86\delta.$$

According to Lemma 2.36,  $r_1$  and  $r_2$  generate a non-elementary subgroup. On the other hand, their translation lengths in  $\lambda X$  are at most  $1000\delta$ , thus

$$\begin{aligned} \operatorname{diam}(Y_{r_1}^{+5\delta} \cap Y_{r_2}^{+5\delta}) &\leq A(G, \lambda X) + 86\delta \leq \lambda A(G, X) + 86\lambda\delta_1 \\ &\leq \lambda(2\pi \operatorname{sh}(10^4 \delta_1) + 86\delta_1). \end{aligned}$$

Thus by (19),  $\Delta(\mathcal{Q}) \leq \Delta_0$ . Let us focus now on  $T(\mathcal{Q})$ . The injectivity radius of  $G$  on  $\lambda X$  is bounded below as follows

$$r_{\operatorname{inj}}(G, \lambda X) \geq \lambda \frac{20\sqrt{\rho_0\delta_1}}{\sqrt{n_1}} \geq \frac{4\pi \operatorname{sh} \rho_0}{n_1} \geq \frac{4\pi \operatorname{sh} \rho_0}{n}$$

In particular for every  $r \in R$  we have  $[r^n]^\infty = n[r]^\infty \geq 4\pi \operatorname{sh} \rho_0$ . Hence  $T(\mathcal{Q}) \geq 4\pi \operatorname{sh} \rho_0$ .  $\square$



On account of the previous lemma, we can now apply the small cancellation theorem (Theorem 6.11) to the action of  $G$  on the rescaled space  $\lambda X$  and the family  $\mathcal{Q}$ . We denote by  $\tilde{X}$  the space obtained by attaching on  $\lambda X$  for every  $(\langle r^n \rangle, Y) \in \mathcal{Q}$ , a cone of radius  $\rho_0$  over the set  $Y_r$ . The quotient of  $\tilde{X}$  by  $K$  is the space  $\bar{X}$ . According to Theorem 6.11,  $\bar{X}$  is geodesic proper and  $\delta_1$ -hyperbolic. Moreover,  $\bar{G}$  acts properly co-compactly by isometries on  $\bar{X}$ . We now prove that the action of  $\bar{G}$  on  $\bar{X}$  also satisfies Points (i)-(iv). Note that the family  $\mathcal{Q}$  satisfies the additional assumptions of Section 6.3.

**Lemma 7.3.** *Every elementary subgroup of  $\bar{G}$  is cyclic either infinite or with order dividing  $n$ .*

*Proof.* By Proposition 6.18, every elementary subgroup of  $\bar{G}$  is cyclic. Let  $\bar{F}$  be a finite subgroup of  $\bar{G}$ . According to Proposition 6.12,  $\bar{F}$  is either isomorphic to a finite subgroup of  $G$  or contained in some  $\text{Stab}(Y_r) / \langle r^n \rangle$  where  $r \in R$ . By Assumption (i) every finite subgroup of  $G$  has order dividing  $n$ . On the other hand  $\text{Stab}(Y_r) / \langle r^n \rangle$  is isomorphic to  $\mathbf{Z}/n\mathbf{Z}$ . Therefore in both cases the order of  $\bar{F}$  divides  $n$ .  $\square$

**Lemma 7.4.** *The constant  $A(\bar{G}, \bar{X})$  is bounded above by  $2\pi \text{sh}(10^4 \delta_1)$  whereas  $r_{\text{inj}}(\bar{G}, \bar{X})$  is bounded below by  $20\sqrt{\rho_0 \delta_1 / n_1}$ .*

*Proof.* We start with the upper bound of  $A(\bar{G}, \bar{X})$ . According to Proposition 6.15,  $A(\bar{G}, \bar{X}) \leq A(G, \lambda X) + \pi \text{sh}(10^4 \delta_1)$ . However the inequality (19) gives

$$A(G, \lambda X) = \lambda A(G, X) \leq 2\lambda\pi \text{sh}(10^4 \delta_1) \leq \pi \text{sh}(10^4 \delta_1).$$

Thus  $A(\bar{G}, \bar{X})$  is bounded above by  $2\pi \text{sh}(10^4 \delta_1)$ . We now focus on the injectivity radius of  $\bar{G}$ . Let  $g$  be a hyperbolic isometry of  $G$  which does not belong to the stabilizer of  $Y_r$  where  $r \in R$ . Its asymptotic translation length in  $\lambda X$  is larger than  $400\lambda\delta_1$ . Proposition 6.13 combined with (20) gives

$$r_{\text{inj}}(\bar{G}, \bar{X}) \geq \min \left\{ \frac{100\lambda\rho_0\delta_1}{\pi \text{sh} \rho_0}, \delta_1 \right\} = \frac{100\lambda\rho_0\delta_1}{\pi \text{sh} \rho_0} = 20\sqrt{\frac{\rho_0\delta_1}{n_1}}. \quad \square$$

**Lemma 7.5.** *The group  $\bar{G}$  is non-elementary. Moreover, there exist  $\bar{u}, \bar{v} \in \bar{G}$  and  $\bar{x} \in \bar{X}$  such that*

$$0 < |\bar{u}\bar{v}\bar{u}^{-1}\bar{v}^{-1}\bar{x} - \bar{x}| \leq \rho_0/20.$$

*Proof.* We denote by  $\bar{u}$  and  $\bar{v}$  the images of  $u$  and  $v$  (see Point (iv)) in  $\bar{G}$  and by  $\bar{x}$  the image of  $x$  in  $\bar{X}$ . In the rescaled space  $\lambda X$  we have

$$0 < |uvu^{-1}v^{-1}x - x| \leq \lambda\rho_0/20 < \rho_0/20.$$

The conclusion follows from Proposition 6.19.  $\square$

Lemmas 7.3, 7.4, 7.5 prove that  $(\bar{G}, \bar{X})$  satisfies the induction hypotheses for exponent  $n$ . The only remaining fact to prove concerns the comparison of the translation lengths.

**Lemma 7.6.** *For every  $g \in G$ , we have*

$$[\bar{g}]_{\bar{X}}^\infty \leq \frac{1}{\sqrt{n_1}} \left( \frac{\pi \text{sh} \rho_0}{5\sqrt{\rho_0 \delta_1}} \right) [g]_X^\infty.$$



*Proof.* Let  $g \in G$ . The asymptotic translation length of  $g$  in the rescaled space  $\lambda X$  is  $[g]_{\lambda X}^\infty = \lambda[g]_X^\infty$ . On the other hand the map  $\lambda X \rightarrow \bar{X}$  shortens the distances, thus  $[\bar{g}]_{\bar{X}}^\infty \leq \lambda[g]_X^\infty$ .  $\square$

This last lemma completes the proof of Proposition 7.1.  $\square$

**Theorem 7.7.** *Let  $G$  be a non-elementary torsion-free hyperbolic group. There exists a critical exponent  $n_0$  such that for every odd integer  $n \geq n_0$ , the quotient  $G/G^n$  is infinite.*

*Proof.* We are actually going to prove that  $G/G^n$  is not finitely presented and therefore infinite. The main ideas are the followings. Using Proposition 7.1 we construct by induction a sequence of groups  $G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow \dots$  where  $G_{k+1}$  is obtained from  $G_k$  by adding new relations of the form  $r^n$ . Then we prove that the direct limit of these groups is exactly  $G/G^n$ . The group  $G/G^n$  cannot be finitely presented otherwise the sequence  $(G_k)$  should be ultimately constant. The parameters  $\rho_0$ ,  $\delta_1$  and  $n_0$  are the one given by Proposition 7.1.

**Critical exponent.** The group  $G$  is torsion-free non-elementary and hyperbolic. Let  $X$  be the Cayley graph of  $G$  with respect to some finite generating set  $S$  of  $G$ . It is a proper geodesic  $\delta$ -hyperbolic space for some  $\delta$  depending on  $S$ . Moreover,  $A(G, X)$  is finite and there exist two elements  $u, v \in G$  which do not commute. In particular if  $x$  is the vertex of  $X$  corresponding to the identity then  $|uvu^{-1}v^{-1}x - x| > 0$ . By rescaling if necessary the space  $X$  we can assume the followings

- (i)  $\delta \leq \delta_1$ ,
- (ii)  $A(G, X) \leq 2\pi \operatorname{sh}(10^4 \delta_1)$ ,
- (iii)  $0 < |uvu^{-1}v^{-1}x - x| \leq \rho_0/20$ .

According to Proposition 2.38,  $r_{inj}(G, X) > 0$ . Therefore, there exists an integer  $n_1 \geq n_0$  such that  $r_{inj}(G, X) \geq 20\sqrt{\rho_0 \delta_1 / n_1}$  and the constant  $c$  defined below is less than 1.

$$c = \frac{1}{\sqrt{n_1}} \left( \frac{\pi \operatorname{sh} \rho_0}{5\sqrt{\rho_0 \delta_1}} \right).$$

From now on we fix an odd integer  $n \geq n_1$ .

**Initialization.** We put  $G_0 = G$  and  $X_0 = X$ . In particular  $(G_0, X_0)$  satisfies the induction hypotheses for exponent  $n$ .

**Induction.** We assume that we already constructed the group  $G_k$  and the space  $X_k$  such that  $(G_k, X_k)$  satisfies the induction hypotheses for exponent  $n$ . We denote by  $R_k$  the set of hyperbolic elements  $r$  of  $G_k$  which are not a proper power and such that  $[r]_{X_k} \leq 1000\delta_1$ . Let  $K_k$  be the normal subgroup of  $G_k$  generated by  $\{r^n, r \in R_k\}$  and  $G_{k+1}$  the quotient of  $G_k$  by  $K_k$ . By Proposition 7.1, there exists a metric space  $X_{k+1}$  such that  $(G_{k+1}, X_{k+1})$  satisfies the induction hypotheses for exponent  $n$ . Moreover, for every  $g \in G_k$ , if we still denote by  $g$  its image in  $G_{k+1}$  we have  $[g]_{X_{k+1}}^\infty \leq c[g]_{X_k}^\infty$ .

**Direct limit.** We denote by  $G_\infty$  the direct limit of the sequence  $(G_k)$ . We claim that  $G_\infty$  is isomorphic to  $G/G^n$ .

Let  $g$  be an element of  $G$ . To shorten the notation we will still denote by  $g$  its image in  $G/G^n$ ,  $G_k$  or  $G_\infty$ . It follows from the construction of the sequence  $(G_k)$  that for every  $k \in \mathbf{N}$  we have  $[g]_{X_k}^\infty \leq c^k [g]_X^\infty$ . Recall that  $c < 1$ . There exists an integer  $k$  such that  $[g]_{X_k}^\infty < 20\sqrt{\rho_0 \delta_1 / n_1}$ . The injectivity radius of  $G_k$  on  $X_k$  is at least  $20\sqrt{\rho_0 \delta_1 / n_1}$ . As an element of  $G_k$  the isometry  $g$  has finite order. However the order of every finite subgroup of  $G_k$  divides  $n$  hence  $g^n$  is trivial in  $G_k$  and thus in  $G_\infty$ . Hence  $G^n$  is contained in the kernel of the map  $G \twoheadrightarrow G_\infty$ .

On the other hand, for every  $k \in \mathbf{N}$ , the kernel of the map  $G_k \rightarrow G_{k+1}$  is generated by the  $n$ -th power of some elements of  $G_k$ . It follows that the kernel of the map  $G \twoheadrightarrow G_\infty$  is exactly  $G^n$ . Consequently,  $G_\infty$  and  $G/G^n$  are isomorphic.

**Conclusion.** Assume now that  $G/G^n$  is finitely presented. Let  $\langle S|R \rangle$  be a finite presentation of  $G/G^n$ . We still denote by  $S$  a preimage of  $S$  in  $G$ . The set  $S$  does not necessarily generate  $G$ , however since  $G$  is also finitely generated there exists  $k_0 \in \mathbf{N}$  such that for all  $k \geq k_0$ ,  $S$  generates  $G_k$ . On the other hand  $R$  is finite, therefore there exists  $k \geq k_0$  such that all the relations in  $R$  are satisfied in the group  $G_k$ . Hence  $G_k$  and  $G/G^n$  are isomorphic. In particular every element of  $G_k$  has finite order. This contradicts the fact that  $G_k$  is a non-elementary hyperbolic group. Consequently,  $G/G^n$  is not finitely presented and thus not finite.  $\square$

## 7.2 A few words about Gromov's monster group

Iterated small cancellation theory can be used to produce many examples of groups with pathological properties, The Gromov monster is such an example [14]. It is built in such a way that its Cayley graph coarsely contains an expander. This is an obstruction for the group to coarsely embed in a Hilbert space. In particular, it cannot satisfy the Baum-Connes conjecture with coefficients.

The global strategy is the following. One starts with a group  $G_0$  satisfying Kazhdan's property (T) and an infinite family of graphs  $(\Theta_k)$ . Then, one constructs a sequence of groups

$$G_0 \twoheadrightarrow G_1 \twoheadrightarrow \cdots \twoheadrightarrow G_k \twoheadrightarrow G_{k+1} \twoheadrightarrow \cdots,$$

where  $G_{k+1}$  is a small cancellation quotient of  $G_k$  in which  $\Theta_k$  is quasi-isometrically embedded. Moreover the previously embedded graphs  $\Theta_\ell$  with  $\ell < k$  are still quasi-isometrically embedded in  $G_{k+1}$  but with different parameters. It uses an extension of graphical small cancellation as sketched in the introduction (Example (ii)). The Gromov monster group is the direct limit of the groups  $G_k$ .

This construction involves many different tools such as small cancellation theory, probability and harmonic analysis, arithmetics,... In this section we only give a few keys to understand the part relying on small cancellation. In particular we will not explained how the graphs  $(\Theta_k)$  need to be chosen and labelled so that they can be used to iterate small cancellation. For the rest of the proof we refer the reader to Gromov's original paper [14] or [1]. In this section we follow with little adaptations the presentation given by G. Arzhantseva and T. Delzant in [1].

We begin by recalling some additional facts about hyperbolic geometry.

**Definition 7.8.** Let  $L \geq 0$ ,  $k \geq 1$  and  $l \geq 0$ . A map  $f : X_1 \rightarrow X_2$  between two metric spaces is an  $L$ -locally  $(k, l)$ -quasi-isometric embedding if for every  $x, x' \in X_1$  such that  $|x - x'| \leq L$ ,

$$k^{-1} |x - x'| - l \leq |f(x) - f(x')| \leq k |x - x'| + l.$$

If  $L = +\infty$  we say that  $f$  is a (globally)  $(k, l)$ -quasi-isometric embedding.

In particular a (locally) quasi-isometric embedding of an interval of  $\mathbf{R}$  is a (local) quasi-geodesic. The following proposition is a consequence of the stability of quasi-geodesics.

**Proposition 7.9.** Let  $\delta \geq 0$ . Let  $k \geq 1$  and  $l \geq 0$ . There exist positive constants  $L$  and  $\alpha$  with the following property. Let  $X_1$  and  $X_2$  be two geodesic metric spaces such that  $X_2$  is  $\delta$ -hyperbolic. If  $f : X_1 \rightarrow X_2$  is an  $L$ -locally  $(k, l)$ -quasi-isometric embedding then it is a (globally)  $(2k, l)$ -quasi-isometric embedding and its image  $f(X_1)$  in  $X_2$  is  $\alpha$ -quasi-convex.

**Remark.** In this proposition the constants  $L$  and  $\alpha$  only depend on  $k$ ,  $\delta$  and  $l$ . Using a rescaling argument we obtain that the best possible values for  $L$  and  $\alpha$  satisfy the following property. For every  $\lambda > 1$ ,  $L(k, \lambda l, \lambda \delta) = \lambda L(k, l, \delta)$  and  $\alpha(k, \lambda l, \lambda \delta) = \lambda \alpha(k, l, \delta)$ . In particular  $\alpha$  tends to 0 as  $\delta$  and  $l$  approach 0.

We present now one step of the construction, namely how the quotient  $G_{k+1}$  is built from  $G_k$ . This operation should be thought as a partial analogue for the Gromov monster group of Proposition 7.1. From now on,  $G$  is a torsion-free non-elementary hyperbolic group and  $X$  its Cayley graph with respect to some finite generating set  $S$  of  $G$ . We assume here that  $S \cap S^{-1} = \emptyset$ . Let  $\Theta$  be a finite connected graph with no vertex of degree 1. A labeling of  $\Theta$  by  $S$  is a map  $m$  which assigns to each edge of  $\Theta$  an orientation and a letter of  $S$ . If we fix a base point in  $\Theta$  then  $m$  induces a homomorphism  $m_*$  from  $\pi_1(\Theta)$  to  $G$  defined as follows. Given a simplicial loop  $\gamma$  in  $\Theta$ , its image under  $m_*$  is the element of  $G$  represented by the word over the alphabet  $S \cup S^{-1}$  that labels  $\gamma$ . Let  $T$  be the universal cover of  $\Theta$ . The labeling  $m$  also induces a  $\pi_1(\Theta)$ -equivariant simplicial map from  $T$  to  $X$  that we denote by  $f$ . We write  $Y$  for the image under  $f$  of  $T$  in  $X$  and  $H$  for the image under  $m_*$  of  $\pi_1(\Theta)$  in  $G$ .

The goal is to understand the quotient  $\bar{G} = G / \langle\langle H \rangle\rangle$  where  $\langle\langle H \rangle\rangle$  is the normal subgroup generated by  $H$ . In particular we want to prove that it is a hyperbolic group. To that end we introduce the following family.

$$\mathcal{Q} = \left\{ (gHg^{-1}, gY) \mid g \in G \right\}.$$

Note that if  $g$  belongs to  $H$  then  $(gY, gHg^{-1})$  and  $(Y, H)$  defines the same pair. However since we consider  $\mathcal{Q}$  as a set of pairs, it will count only once. We define then two small cancellation parameters which respectively play the role of the length of largest piece and the length of the smallest relation in the usual small cancellation theory.

$$\begin{aligned} \Delta'(\mathcal{Q}) &= \sup_{g \in G \setminus H} \text{diam}(Y^{+12\delta} \cap gY^{+12\delta}) \\ T(\mathcal{Q}) &= \inf \{[h] \mid h \in H \setminus \{1\}\} \end{aligned}$$

Note that the parameter  $\Delta'(\mathcal{Q})$  is not exactly the same as the one we used before. Indeed the regular parameter  $\Delta(\mathcal{Q})$  would be

$$\Delta(\mathcal{Q}) = \sup_{g \in G \setminus (\text{Stab}(Y) \cap \text{Norm}(H))} \text{diam}(Y^{+12\delta} \cap gY^{+12\delta}),$$

where  $\text{Norm}(H)$  is the normalizer of  $H$  in  $G$ . In particular  $\Delta(\mathcal{Q}) \leq \Delta'(\mathcal{Q})$ . The small cancellation assumptions that will follow will be thus stronger than the one we presented. As an other consequence we can see that if  $\Delta'(\mathcal{Q})$  is finite then  $H$  is the whole stabilizer of  $Y$ . The small cancellation theorem used by G. Arzhantseva and T. Delzant in [1] can be stated as follows.

**Theorem 7.10** (Compare [1, Th. 3.10]). *There exists  $\rho_0 \geq 10^{20}\delta$  such that for all  $k \geq 1$ , for all  $\rho \geq \rho_0$ , there exist positive numbers  $\delta_2$  and  $\Delta_2$  which do not depend on  $G$ ,  $\Theta$  or  $m$  with the following property. Let  $l \geq 0$ . Let  $L = L(k, l, \delta)$  and  $\alpha = \alpha(k, l, \delta)$  be the constants given by Proposition 7.9. Assume that  $f : T \rightarrow X$  is an  $L$ -locally  $(k, l)$ -quasi-isometric embedding. If*

$$\frac{\delta}{T(\mathcal{Q})} \leq \delta_2, \quad \frac{\alpha}{T(\mathcal{Q})} \leq 10\delta_2 \quad \text{and} \quad \frac{\Delta'(\mathcal{Q})}{T(\mathcal{Q})} \leq \Delta_2,$$

*then the followings hold.*

- (i) *The quotient  $\bar{G}$  is a non-elementary torsion-free hyperbolic group. Moreover the hyperbolicity constant of its Cayley graph with respect to the image of  $S$  in  $\bar{G}$  can be bounded by a number only depending on  $\rho$ ,  $\delta$ ,  $T(\mathcal{Q})$  and  $\text{diam } \Theta$ .*
- (ii) *The projection  $G \rightarrow \bar{G}$  induces an embedding from  $B(1, R)$  in  $\bar{G}$  where  $B(1, R)$  is the ball of  $G$  of center 1 and radius*

$$R = \frac{1}{20} \cdot \frac{\rho}{\pi \text{sh } \rho} T(\mathcal{Q}).$$

- (iii) *The map  $f$  induces a map (that we still denote  $f$ ) from  $\Theta$  into the Cayley graph of  $\bar{G}$  satisfying for all  $x, x' \in \Theta$*

$$|f(x) - f(x')| \geq \frac{1}{500} \cdot \frac{T(\mathcal{Q})}{\text{diam } \Theta} \cdot \frac{\rho}{\pi \text{sh } \rho} \left( \frac{1}{2k} |x - x'| - l \right).$$

*Sketch of proof.* We denote by  $\rho_0$ ,  $\delta_0$  and  $\Delta_0$  the constants given by Theorem 6.11. Without loss of generality we can assume that  $\rho_0 \geq 10^{20}\delta$ ,  $\delta_0 \leq 10^{-10}\delta$  and  $\Delta_0 \geq 100\delta_0$ . Let  $k \geq 1$  and  $\rho \geq \rho_0$ . We put

$$\delta_2 = \frac{\delta_0}{\pi \text{sh } \rho} \quad \text{and} \quad \Delta_2 = \frac{\Delta_0 - 28\delta_0}{\pi \text{sh } \rho}$$

Since  $f : T \rightarrow X$  is a  $L$ -locally  $(k, l)$ -quasi-isometric embedding in  $X$  it follows from Proposition 7.9 that  $f$  is actually a  $(2k, l)$ -quasi-isometric embedding in  $X$  and  $Y = f(T)$  is  $\alpha$ -quasi-convex.

We now consider the action of  $G$  on the rescaled space  $\lambda X$  where  $\lambda = \pi \text{sh } \rho / T(\mathcal{Q})$ . In particular  $\lambda X$  is  $\delta_0$ -hyperbolic and for every  $h \in H \setminus \{1\}$ ,  $[h]_{\lambda X} \geq \pi \text{sh } \rho$ . Moreover  $Y$ , viewed as a subset of  $\lambda X$  is  $10\delta_0$ -quasi-convex. Note that  $Y$  is not necessarily strongly-quasi-convex. We denote by

$Z$  the  $12\delta_0$ -neighborhood of  $Y$  in  $\lambda X$ . By Proposition 2.21,  $Z$  is strongly quasi-convex. Instead of working with  $\mathcal{Q}$  we consider the family  $\mathcal{S}$  defined as

$$\mathcal{S} = \left\{ (gHg^{-1}, gZ) \mid g \in G \right\}.$$

According to Lemma 2.13,  $\Delta'(\mathcal{S})$  is bounded above by  $\lambda\Delta'(\mathcal{Q}) + 28\delta_0$ . Hence  $\Delta(\mathcal{S}) \leq \Delta'(\mathcal{S}) \leq \Delta_0$ . On the other hand  $T(\mathcal{S}) = \lambda T(\mathcal{Q}) \geq \pi \operatorname{sh} \rho$ . Thus  $\mathcal{S}$  satisfies the assumptions of Theorem 6.11. Let  $\dot{X}$  be the cone-off obtain by attaching for every  $g \in G/H$  a cone of radius  $\rho$  and base  $Z$  on  $X$ . Let  $\bar{X}$  be the quotient of  $\dot{X}$  by  $\langle\langle H \rangle\rangle$ . By Theorem 6.11,  $\bar{X}$  is a proper geodesic hyperbolic space and  $\bar{G}$  acts properly co-compactly by isometries on it. In particular  $\bar{G}$  is hyperbolic.

Recall that the embedding  $\lambda X \hookrightarrow \dot{X}$  is 1-Lipschitz. On the other hand the map  $\nu : \dot{X} \rightarrow \bar{X}$  induces an isometry from  $B(1, \rho/20)$  onto its image (see Proposition 3.15). It follows that the projection  $G \rightarrow \bar{G}$  induces an embedding from  $B(1, R)$  in  $\bar{G}$  where

$$R = \lambda^{-1} \frac{\rho}{20} = \frac{1}{20} \frac{\rho}{\pi \operatorname{sh} \rho} T(\mathcal{Q}).$$

This proves Point (ii). Similar techniques provide an estimate of the hyperbolicity constant of  $\bar{G}$  and Point (iii). See [1, Section 3.4] for the details.

**Remark.** By assumption  $\rho \geq 10^{20}\delta$  and

$$\frac{\delta}{T(\mathcal{Q})} \leq \delta_2 = \frac{\delta_0}{\pi \operatorname{sh} \rho} \leq \frac{10^{-10}\delta}{\pi \operatorname{sh} \rho}.$$

It follows that

$$R = \frac{1}{20} \frac{\rho}{\pi \operatorname{sh} \rho} T(\mathcal{Q}) \geq \frac{10^{200}\delta}{20}.$$

In particular  $R$  is very large compare to the hyperbolicity constant  $\delta$  of  $X$ , the Cayley graph of  $G$ . Thus Point (ii) is far from being a vacuous observation.

We now prove that  $\bar{G}$  is torsion-free and non-elementary. By assumption  $G$  is torsion-free. It follows from Proposition 6.12 that  $\bar{G}$  is torsion free if and only if  $\operatorname{Stab}(Y)/H$  is. The small cancellation assumption involving  $\Delta'(\mathcal{Q})$  is much stronger than the regular assumption of Theorem 6.11. As we already noticed, since  $\Delta'(\mathcal{Q})$  is finite the subgroup  $H$  is exactly  $\operatorname{Stab}(Y)$ . Hence  $\bar{G}$  is torsion-free. In particular every elementary subgroup of  $\bar{G}$  is cyclic. Assume now that  $\bar{G}$  is elementary. Then  $\bar{G}$  would be cyclic thus abelian. In particular, all the generators of  $S$  would commute in  $\bar{G}$ . This is not compatible with the embedding of  $B(1, R)$  into  $\bar{G}$ . Hence  $\bar{G}$  is not elementary.  $\square$

This theorem can be now used to iterate the small cancellation process. One has in some sense more freedom than for Burnside groups. In order to uniformly bound the exponent during the induction we had indeed to provide precise estimates of  $A(\bar{G}, \bar{X})$ ,  $r_{inj}(\bar{G}, \bar{X})$ , etc. This is not needed here. In particular if the girth of the next graph one wants to embed is very large compare to the previous parameters then it will be easy to satisfy again the assumptions  $\delta/T(\mathcal{Q}) \leq \delta_2$  and  $\alpha/T(\mathcal{Q}) \leq 10\delta_2$ . In this way the group that we obtain at the end is lacunary hyperbolic. The difficulty here is to exhibit the right family of graphs and an appropriate labeling so that one can repeat the process.

## A Appendix: Cartan-Hadamard Theorem

In [10] the authors state and exploit a Cartan-Hadamard theorem for orbifolds. They prove that if an orbifold is locally hyperbolic (with the appropriate quantifiers) then it is developable and its universal cover (in the sense of orbifold) is globally hyperbolic. As we explained, rotation families provide a means for avoiding orbifolds. However we still need a “regular” Cartan-Hadamard theorem. This result was given first in Gromov’s original paper about hyperbolic groups [12]. Other proofs using isoperimetric inequalities can be found in [3, Chap. 8, Th. 8.1.2] and [26]. In this appendix we present an other proof based on topology. It relies on the ideas used by T. Delzant and M. Gromov for the case of orbifolds.

Let  $X$  be a length space. We say that  $X$  is  $\sigma$ -locally  $\delta$ -hyperbolic if every ball of radius  $\sigma$  is  $\delta$ -hyperbolic i.e., for every four points  $x, y, z$  and  $t$  contained in such a ball

$$\langle x, z \rangle_t \geq \min \{ \langle x, y \rangle_t, \langle y, z \rangle_t \} - \delta.$$

If its fundamental group is normally generated by free homotopy classes of loops of diameter less than  $r$  then  $X$  is said to be  $r$ -simply connected. In particular, if  $X$  is simply connected, it is  $r$ -simply connected for every  $r > 0$ .

**Theorem A.1** (Cartan-Hadamard Theorem). *Let  $\delta \geq 0$  and  $\sigma > 10^7 \delta$ . Let  $X$  be a length space. If  $X$  is  $\sigma$ -locally  $\delta$ -hyperbolic and  $10^{-5} \sigma$ -simply connected then  $X$  is (globally)  $300\delta$ -hyperbolic.*

**General strategy.** Let us first give the main ideas of the proof. We are going to demonstrate a version of stability of local quasi-geodesics in  $X$  (Proposition A.13). The global hyperbolicity will follow from it (Proposition A.19). To that end we endow the set of paths of  $X$  with a binary relation (Definition A.3). Two related paths notably fellow travel and have the same endpoints. Restricted to the set of local quasi-geodesics, this relation turns out to be an equivalence relation (Proposition A.8). After fixing a base point  $x_0$  in  $X$ , we look at the set  $\tilde{X}$  of equivalence classes of local quasi-geodesics starting at  $x_0$ . It comes with a natural map  $p : \tilde{X} \rightarrow X$  which sends each path to its terminal point. We show that, endowed with the appropriate topology,  $\tilde{X}$  is a path-connected cover of  $X$  (Lemma A.15 and Proposition A.18). If  $X$  was simply connected this would force  $p$  to be a bijection. Actually it remains true if  $X$  is just  $10^{-5} \sigma$ -simply connected. It implies that two paths starting at  $x_0$  with the same terminal points are equivalent and therefore at bounded Hausdorff distance.

Before starting the proof we recall a well-known fact about hyperbolic space. If  $X$  is a geodesic hyperbolic space the distance function is quasi-convex [6, Chap. 10, Cor. 5.3]. The next proposition gives a precise statement of the analog property for two quasi-geodesics in a hyperbolic length space.

**Proposition A.2.** *Let  $X$  be a  $\delta$ -hyperbolic length space. Let  $\gamma : [a, b] \rightarrow X$  and  $\gamma' : [a', b'] \rightarrow X$  be respectively  $(1, l)$ - and  $(1, l')$ -quasi-geodesics. Let  $f : [0, 1] \rightarrow \mathbf{R}$  be the function defined by*

$$f(\theta) = \left| \gamma \left( (1 - \theta)a + \theta b \right) - \gamma' \left( (1 - \theta)a' + \theta b' \right) \right|$$

For every  $\theta \in [0, 1]$ ,

$$f(\theta) \leq \theta f(0) + (1 - \theta) f(1) + 2l + 2l' + 8\delta$$

*Proof.* Let us first examine the case where  $\gamma$  and  $\gamma'$  start at the same point. We put  $x = \gamma(a) = \gamma'(a')$ . Let  $y$  and  $y'$  be the respective endpoints of  $\gamma$  and  $\gamma'$ . Let  $\theta \in [0, 1]$ . The points  $s$  and  $s'$  respectively stand for  $\gamma((1-\theta)a + \theta b)$  and  $\gamma'((1-\theta)a' + \theta b')$ . Since  $\gamma$  is a  $(1, l)$ -quasi-geodesic  $s$  satisfies  $\langle x, y \rangle_s \leq l/2$ . Moreover,

$$\theta |x - y| - l \leq |x - s| \leq \theta |x - y| + l.$$

The same kind of properties holds for  $s'$ . By Lemma 2.2-(iii) we have

$$|s - s'| \leq \max \left\{ \left| |x - s| - |x - s'| \right| + \max \{l, l'\}, |x - s| + |x - s'| - 2 \langle y, y' \rangle_x \right\} + 4\delta.$$

However using the triangle inequality we have

$$\left| |x - s| - |x - s'| \right| \leq \theta \left| |x - y| - |x - y'| \right| + l + l' \leq \theta |y - y'| + l + l'.$$

On the other hand,

$$\begin{aligned} |x - s| + |x - s'| - 2 \langle y, y' \rangle_x &\leq \theta \left( |x - y| + |x - y'| \right) - 2 \langle y, y' \rangle_x + l + l' \\ &\leq \theta |y - y'| - 2(1 - \theta) \langle y, y' \rangle_x + l + l'. \end{aligned}$$

Combining these inequalities we obtain that  $f(\theta) = |s - s'|$  is bounded above by

$$\theta |y - y'| + l + l' + \max \{l, l'\} + 4\delta = \theta f(1) + l + l' + \max \{l, l'\} + 4\delta.$$

Let us consider now the case where  $\gamma$  and  $\gamma'$  have different extremities. We denote by  $x$  and  $y$  (respectively  $x'$  and  $y'$ ) the endpoints of  $\gamma$  (respectively  $\gamma'$ ) such that  $f(0) = |x - x'|$  and  $f(1) = |y - y'|$ . Let  $\eta > 0$ . There exists a  $(1, \eta)$ -quasi-geodesic  $\mu : [c, d] \rightarrow X$  such that  $\mu(c) = x$  and  $\mu(d) = y'$ . Let  $\theta \in [0, 1]$ . The points  $s$ ,  $t$  and  $s'$  respectively stand for  $\gamma((1-\theta)a + \theta b)$ ,  $\mu((1-\theta)c + \theta d)$  and  $\gamma'((1-\theta)a' + \theta b')$ . Hence  $f(\theta) = |s - s'| \leq |s - t| + |t - s'|$ . It follows from the previous case that

$$\begin{aligned} |s - t| &\leq \theta |y - y'| + l + \eta + \max \{l, \eta\} + 4\delta, \\ |t - s'| &\leq (1 - \theta) |x - x'| + \eta + l' + \max \{\eta, l'\} + 4\delta. \end{aligned}$$

Thus

$$f(\theta) \leq (1 - \theta)f(0) + \theta f(1) + l + l' + 2\eta + \max \{l, \eta\} + \max \{\eta, l'\} + 8\delta.$$

This last inequality holds for every  $\eta > 0$  which completes the proof.  $\square$

For the rest of the appendix we fix  $\delta \geq 0$  and  $\sigma > 10^7 \delta$ . We denote by  $X$  a  $\sigma$ -locally  $\delta$ -hyperbolic and  $10^{-5}\sigma$ -simply connected length space.

### A.1 Following paths. Definition and first properties.

We recall that the paths that we consider are rectifiable and parametrized by arc length. The length of a path  $\gamma$  is denoted by  $L(\gamma)$ .

**Definition A.3.** Let  $D \geq 0$ . Let  $\gamma : [a, b] \rightarrow X$  and  $\gamma' : [a', b'] \rightarrow X$  be two paths of  $X$ . We say that  $\gamma$   $D$ -follows  $\gamma'$  if there exists a non decreasing map  $\theta : [a, b] \rightarrow [a', b']$  satisfying the following properties.

- (i)  $\theta(a) = a'$  and  $\theta(b) = b'$ ,
- (ii) for every  $s \in [a, b]$ ,  $|\gamma(s) - \gamma' \circ \theta(s)| < D$ ,
- (iii) for every  $s, t \in [a, b]$ , if  $|s - t| \leq \sigma/4$  then  $|\theta(s) - \theta(t)| \leq \sigma/3$ .

If  $D = 10^{-2}\sigma$  we simply say that  $\gamma$  follows  $\gamma'$ .

**Remarks.** We keep the notations of the definition and assume that  $\gamma$   $D$ -follows  $\gamma'$ . Let  $s, t \in [a, b]$  such that  $s \leq t$ . The restriction of  $\gamma$  to  $[s, t]$  also  $D$ -follows the restriction of  $\gamma'$  to  $[\theta(s), \theta(t)]$ . If  $\gamma'$  is a  $\sigma/3$ -locally  $(1, l')$ -quasi-geodesic path then for all  $s, t \in [a, b]$  such that  $|s - t| \leq \sigma/4$ ,

$$\begin{aligned} |\theta(s) - \theta(t)| &\leq |\gamma' \circ \theta(s) - \gamma' \circ \theta(t)| + l' \\ &\leq |\gamma' \circ \theta(s) - \gamma(s)| + |\gamma(s) - \gamma(t)| + |\gamma(t) - \gamma' \circ \theta(t)| + l' \\ &\leq |s - t| + 2D + l' \end{aligned}$$

Applying this fact twice we obtain that for every  $s, t \in [a, b]$  such that  $|s - t| \leq \sigma/2$  we have  $|\theta(s) - \theta(t)| \leq |s - t| + 4D + 2l'$ . We will use this argument very often.

Every path  $D$ -follows itself. More generally let  $\gamma : [a, b] \rightarrow X$  be a path and  $x, y$  be two points of  $X$  such that  $\max\{|x - \gamma(a)|, |y - \gamma(b)|\} < D$ . There exists a path joining  $x$  to  $y$  which  $D$ -follows  $\gamma$ . This path can be obtained as a concatenation of  $\gamma_x$ ,  $\gamma$  and  $\gamma_y$  where  $\gamma_x$  (respectively  $\gamma_y$ ) is a  $(1, l)$ -quasi-geodesic joining  $x$  to  $\gamma(a)$  (respectively  $\gamma(b)$  to  $y$ ) for a sufficiently small  $l$ .

The next lemmas explain how this notion behaves with respect to the concatenation and extension of paths.

**Lemma A.4.** Let  $l' \in (0, 10^{-5}\sigma)$  and  $D \in (0, 10^{-2}\sigma)$ . Let  $\gamma : [a, c] \rightarrow X$  be a path and  $\gamma' : [a', c'] \rightarrow X$  be a  $\sigma/3$ -local  $(1, l')$ -quasi-geodesic. Let  $b \in [a, c]$  and  $b' \in [a', c']$ . We assume that the restriction of  $\gamma$  to  $[a, b]$  (respectively  $[b, c]$ )  $D$ -follows the restriction of  $\gamma'$  to  $[a', b']$  (respectively  $[b', c']$ ). Then  $\gamma$   $D$ -follows  $\gamma'$  as well.

*Proof.* The restriction of  $\gamma$  to  $[a, b]$   $D$ -follows the one of  $\gamma'$  to  $[a', b']$ . By definition, there exists a map  $\theta_- : [a, b] \rightarrow [a', b']$  satisfying the axioms of Definition A.3. In the same way we have a map  $\theta_+ : [b, c] \rightarrow [b', c']$ . We define  $\theta : [a, c] \rightarrow [a', c']$  such that its restriction to  $[a, b]$  (respectively  $[b, c]$ ) is  $\theta_-$  (respectively  $\theta_+$ ). Let  $s, t \in [a, c]$  such that  $|s - t| \leq \sigma/4$ . If  $s$  and  $t$  both belong to  $[a, b]$  (respectively  $[b, c]$ ), then it follows from the properties of  $\theta_-$  and  $\theta_+$  that  $|\theta(s) - \theta(t)| \leq \sigma/3$ . Assume now that  $s \in [a, b]$  and  $t \in [a', b']$ . In particular,  $|s - b|$  and  $|b - t|$  are at most  $\sigma/4$ . Since  $\gamma'$  is a  $\sigma/3$ -local  $(1, l')$ -quasi-geodesic, we obtain

$$|\theta(s) - \theta(t)| = |\theta_-(s) - \theta_-(b)| + |\theta_+(b) - \theta_+(t)| \leq |s - t| + 4D + 2l' \leq \frac{\sigma}{3}.$$

Hence for all  $s, t \in [a, c]$  if  $|s - t| \leq \sigma/4$  then  $|\theta(s) - \theta(t)| \leq \sigma/3$ . The other axioms of Definition A.3 can be easily checked, hence  $\gamma$   $D$ -follows  $\gamma'$ .  $\square$



**Lemma A.5.** *Let  $l' \in (0, 10^{-5}\sigma)$  and  $D \in (0, 10^{-2}\sigma)$ . Let  $\gamma : [a, b] \rightarrow X$  be a path and  $\gamma' : [a', c'] \rightarrow X$  be a  $\sigma/3$ -locally  $(1, l')$ -quasi-geodesic. Let  $b' \in [a', c']$  such that  $|b' - c'| \leq \sigma/12 - 2D - l'$ . Moreover, we assume that  $|\gamma(b) - \gamma'(c')| < D$ . If  $\gamma$   $D$ -follows the restriction of  $\gamma'$  to  $[a', b']$  then it  $D$ -follows  $\gamma'$  as well.*

*Proof.* The path  $\gamma$   $D$ -follows the restriction of  $\gamma'$  to  $[a', c']$ . This provides a map  $\theta : [a, b] \rightarrow [a', c']$  satisfying the axioms of Definition A.3. We define a map  $\mu : [a, b] \rightarrow [a', b']$  by  $\mu(s) = \theta(s)$  for every  $s \in [a, b]$  and  $\mu(b) = b'$ . Let  $s, t \in [a, c]$ ,  $s \leq t$  such that  $|s - t| \leq \sigma/4$ . If  $t \neq b$ , the fact follows from the properties of  $\theta$  that  $|\mu(s) - \mu(t)| \leq \sigma/3$ . Assume now that  $t = b$ . Since  $\gamma'$  is a  $\sigma/3$ -local  $(1, l')$ -quasi-geodesic, we obtain

$$|\mu(s) - \mu(t)| = |\theta(s) - \theta(b)| + |c' - b'| \leq |s - b| + 2D + l' + |c' - b'| \leq \frac{\sigma}{3}.$$

The other axioms of Definition A.3 can be easily checked, hence  $\gamma$   $D$ -follows  $\gamma'$ .  $\square$

**Remark.** The two previous lemmas lead to the following fact. Let  $l \in (0, 10^{-5}\sigma)$  and  $D \in (0, 10^{-2}\sigma)$ . Let  $\gamma : [a, c] \rightarrow X$  be a  $\sigma/3$ -local  $(1, l)$ -quasi-geodesic. Let  $b \in [a, c]$  such that  $|b - c| < D$ . Then  $\gamma$   $D$ -follows its restriction to  $[a, b]$  and conversely.

## A.2 Transitivity of the relation

One important property of the notion of  $D$ -following lies in this fact: assume that  $\gamma$  follows  $\gamma'$ ; if  $\gamma$  and  $\gamma'$  are quasi-geodesics and have close extremities then  $\gamma$  actually  $D$ -follows  $\gamma'$  with  $D$  much smaller than  $10^{-2}\sigma$ . This assertion is carefully proved in the next two lemmas. It is the crucial point needed to define later a binary relation which is transitive.

**Lemma A.6.** *Let  $l, l' \in (0, 10^{-5}\sigma)$ . Let  $\gamma : [a, b] \rightarrow X$  (respectively  $\gamma : [a', b'] \rightarrow X$ ) be a  $(1, l)$ -quasi-geodesic (respectively  $(1, l')$ -quasi-geodesic). We denote by  $x$  and  $y$  (respectively  $x'$  and  $y'$ ) the initial and terminal points of  $\gamma$  (respectively  $\gamma'$ ). We assume that*

$$\max\{|x - x'|, |y - y'|\} < \sigma/200.$$

*If  $\gamma$  and  $\gamma'$  are contained in a ball of radius  $\sigma$  of  $X$  then the path  $\gamma$   $D$ -follows  $\gamma'$  and conversely where*

$$D = \max\{|x - x'|, |y - y'|\} + 3l + 3l' + 8\delta.$$

*Proof.* The paths  $\gamma$  and  $\gamma'$  are contained in a ball of radius  $\sigma$ . Thus everything works as if we were in a hyperbolic space. We denote by  $\theta$  the increasing affine function which maps  $[a, b]$  onto  $[a', b']$ . In particular,  $\theta(a) = a'$  and  $\theta(b) = b'$ . By Proposition A.2, for every  $t \in [a, b]$  we have

$$|\gamma(t) - \gamma' \circ \theta(t)| < \max\{|x - x'|, |y - y'|\} + 3l + 3l' + 8\delta.$$

On the other hand  $\gamma'$  is a  $(1, l')$ -quasi-geodesic. Hence for every  $s, t \in [a, b]$  such that  $|s - t| \leq \sigma/4$ , we have  $|\theta(s) - \theta(t)| \leq |s - t| + 2D + l' \leq \sigma/3$ . Therefore  $\gamma$   $D$ -follows  $\gamma'$ . The statement is symmetric, thus  $\gamma'$  also  $D$ -follows  $\gamma$ .  $\square$

**Lemma A.7.** *Let  $l, l' \in (0, 10^{-5}\sigma)$ . Let  $\gamma : [a, b] \rightarrow X$  (respectively  $\gamma : [a', b'] \rightarrow X$ ) be a  $\sigma/3$ -local  $(1, l)$ -quasi-geodesic (respectively  $\sigma/3$ -local  $(1, l')$ -quasi-geodesic). We denote by  $x$  and  $y$  (respectively  $x'$  and  $y'$ ) the initial and terminal points of  $\gamma$  (respectively  $\gamma'$ ). We assume that*

$$\max\{|x - x'|, |y - y'|\} < \sigma/200.$$

*If  $\gamma$  follows  $\gamma'$ , then  $\gamma$  actually  $D$ -follows  $\gamma'$  and conversely where*

$$D = \max\{|x - x'|, |y - y'|, l + l' + 5\delta\} + 3l + 3l' + 8\delta.$$

*Proof.* For simplicity of notation we put  $\tau = \sigma/25$ . The path  $\gamma$  follows  $\gamma'$ . Hence there exists a non-decreasing function  $\theta : [a, b] \rightarrow [a', b']$  satisfying the properties of Definition A.3. In particular, for every  $t \in [a, b]$  we have  $|\gamma(t) - \gamma' \circ \theta(t)| < 10^{-2}\sigma$ . The point  $\gamma(t)$  is actually much closer to  $\gamma'$ : we claim that for every  $t \in [a + \tau, b - \tau]$ , there exists  $t' \in [\theta(t - \tau), \theta(t + \tau)]$  such that  $|\gamma(t) - \gamma'(t')| \leq l + l' + 5\delta$ . Note that  $2\tau \leq \sigma/4$ . Therefore  $\gamma$  and  $\gamma'$  restricted to  $[t - \tau, t + \tau]$  and  $[\theta(t - \tau), \theta(t + \tau)]$  are respectively  $(1, l)$ - and  $(1, l')$ -quasi-geodesics contained in a ball  $B$  of radius  $\sigma$ . In particular, the second one is  $(l' + 3\delta)$ -quasi-convex in the  $\delta$ -hyperbolic ball  $B$ . Moreover  $|\gamma(t) - \gamma(t \pm \tau)| \geq \sigma/25 - l$ . It follows from the hyperbolicity of  $B$  that

$$\langle \gamma' \circ \theta(t - \tau), \gamma' \circ \theta(t + \tau) \rangle_{\gamma(t)} \leq \langle \gamma(t - \tau), \gamma(t + \tau) \rangle_{\gamma(t)} + 2\delta \leq l + 2\delta.$$

By quasi-convexity, the distance between  $\gamma(t)$  and the path  $\gamma'$  restricted to  $[\theta(t - \tau), \theta(t + \tau)]$  is less than or equal to  $l + l' + 5\delta$ , which establishes our claim.

Let us now consider the subdivision  $a = t_0 \leq \dots \leq t_m = b$  of  $[a, b]$  such that for all  $i \in \{0, \dots, m - 2\}$ ,  $|t_{i+1} - t_i| = 2\tau$  and  $2\tau \leq |t_m - t_{m-1}| \leq 4\tau$ . First we put  $t'_0 = a'$  and  $t'_m = b'$ . According to our previous claim, for every  $i \in \{1, \dots, m - 1\}$  there exists  $t'_i \in [\theta(t_i - \tau), \theta(t_i + \tau)]$  such that  $|\gamma(t_i) - \gamma'(t'_i)| \leq l + l' + 5\delta$ . It follows that for every  $i \in \{0, \dots, m\}$ ;

$$|\gamma(t_i) - \gamma'(t'_i)| \leq \max\{|x - x'|, |y - y'|, l + l' + 5\delta\} < \sigma/200,$$

Let  $i \in \{1, \dots, m - 2\}$ . By construction  $t_i + \tau = t_{i+1} - \tau$ . Since  $\theta$  is non-decreasing we have

$$\theta(t_i - \tau) \leq t'_i \leq \theta(t_i + \tau) = \theta(t_{i+1} - \tau) \leq t'_{i+1} \leq \theta(t_{i+1} + \tau).$$

Moreover,  $|(t_{i+1} + \tau) - (t_i - \tau)| = 4\tau \leq \sigma/4$ . By definition of  $\theta$ ,

$$|t'_{i+1} - t'_i| \leq |\theta(t_{i+1} + \tau) - \theta(t_i - \tau)| \leq \frac{\sigma}{3}.$$

The same facts hold for  $i = 0$  and  $i = m - 1$ . Therefore  $a' = t'_0 \leq \dots \leq t'_m = b'$  is a subdivision of  $[a', b']$  such that for all  $i \in \{0, \dots, m - 1\}$ ,  $|t'_{i+1} - t'_i| \leq \sigma/3$ . In particular,  $\gamma$  restricted to  $[t_i, t_{i+1}]$  and  $\gamma'$  restricted to  $[t'_i, t'_{i+1}]$  are respectively  $(1, l)$ - and  $(1, l')$ -quasi-geodesics contained in a ball of radius  $\sigma$ . According to Lemma A.6, they  $D$ -follows each other with

$$D = \max\{|x - x'|, |y - y'|, l + l' + 5\delta\} + 3l + 3l' + 8\delta.$$

By concatenation (Lemma A.4),  $\gamma$   $D$ -follows  $\gamma'$  and conversely.  $\square$

**Proposition A.8.** *Let  $l \in (0, 10^{-5}\sigma)$ . For every  $i \in \{1, 2, 3\}$ , let  $\gamma_i : [a_i, b_i] \rightarrow X$  be a  $\sigma/3$ -local  $(1, l)$ -quasi-geodesic joining  $x_i$  to  $y_i$ . We assume that*

$$(i) \max\{|x_1 - x_2|, |x_2 - x_3|, |y_1 - y_2|, |y_2 - y_3|\} < \sigma/400,$$

$$(ii) \gamma_1 \text{ follows } \gamma_2 \text{ and } \gamma_2 \text{ follows } \gamma_3$$

*Then  $\gamma_1$  follows  $\gamma_3$ .*

*Proof.* According to Lemma A.7 and Hypothesis (i),  $\gamma_1$  actually  $\sigma/200$ -follows  $\gamma_2$ . It gives us a map  $\theta_1 : [a_1, b_1] \rightarrow [a_2, b_2]$  satisfying the axioms of Definition A.3. In the same way,  $\gamma_2$   $\sigma/200$ -follows  $\gamma_3$  providing a map  $\theta_2 : [a_2, b_2] \rightarrow [a_3, b_3]$ . We denote by  $\theta : [a_1, b_1] \rightarrow [a_3, b_3]$  the composition  $\theta = \theta_2 \circ \theta_1$ . By triangle inequality, for all  $s \in [a_1, b_1]$ ,

$$|\gamma_1(s) - \gamma_3 \circ \theta(s)| \leq |\gamma_1(s) - \gamma_2 \circ \theta_1(s)| + |\gamma_2 \circ \theta_1(s) - \gamma_3 \circ \theta_2 \circ \theta_1(s)| < 10^{-2}\sigma.$$

Let  $s, t \in [a_1, b_1]$  such that  $|s - t| \leq \sigma/4$ . Since  $\gamma_2$  is a  $\sigma/3$ -local  $(1, l)$ -quasi-geodesic, we have

$$|\theta_1(s) - \theta_1(t)| \leq |s - t| + \frac{\sigma}{50} + l \leq \frac{\sigma}{2}.$$

However  $\gamma_3$  is also a  $\sigma/3$ -local  $(1, l)$ -quasi-geodesic, hence

$$|\theta(s) - \theta(t)| \leq |\theta_1(s) - \theta_1(t)| + \frac{\sigma}{25} + 2l \leq |s - t| + \frac{3\sigma}{50} + 3l \leq \frac{\sigma}{3}.$$

The other properties of Definition A.3 can be easily checked, hence  $\gamma_1$  follows  $\gamma_3$ .  $\square$

**Definition A.9.** Let  $l \in (0, 10^{-5}\sigma)$ . Let  $\gamma : [a, b] \rightarrow X$  and  $\gamma' : [a', b'] \rightarrow X$  be two  $\sigma/3$ -local  $(1, l)$ -quasi-geodesics. We say that  $\gamma$  and  $\gamma'$  are *related* if they have the same endpoints and follow each other.

This binary relation is reflexive and symmetric. By Proposition A.8, it is transitive as well. Therefore it is an equivalence relation.

**Remark.** The equivalence relation used by T. Delzant and M. Gromov in [10] is the following: two paths are equivalent if they have the same extremities and their Hausdorff distance is at most  $200\delta$ . However this formulation is not sufficient to complete the proofs. Indeed, to be able to use the local hyperbolicity of  $X$  one needs to be sure that every small portion of one path is close to a small portion of the other path, which may not be the case with a global assumption on the Hausdorff distance. That is the reason why we introduced the notion of  $D$ -following paths.

### A.3 Existence of following quasi-geodesics

**Proposition A.10.** *Let  $l' \in (0, 10^{-5}\sigma)$ . Let  $x, x', y$  and  $y'$  be four points of  $X$  such that*

$$\max\{|x - x'|, |y - y'|\} < \sigma/200.$$

*Let  $\gamma' : [a', b'] \rightarrow X$  be a  $\sigma/3$ -locally  $(1, l')$ -quasi-geodesic joining  $x'$  to  $y'$ . For every  $l > 0$ , there exists a  $\sigma/3$ -local  $(1, l)$ -quasi-geodesic  $\gamma$  joining  $x$  to  $y$  which follows  $\gamma'$ .*

*Proof.* Let  $l \in (0, 10^{-10}\sigma)$ . We denote by  $\mathcal{C}$  the set of paths joining  $x$  to  $y$  which follow  $\gamma'$ . This set is non-empty. We fix a path  $\gamma : [a, b] \rightarrow X$  of  $\mathcal{C}$  such that  $L(\gamma) \leq \inf_{c \in \mathcal{C}} L(c) + l$ . By construction there is a map  $\theta : [a, b] \rightarrow [a', b']$  satisfying the axioms of Definition A.3. For simplicity of notation, we put  $\tau = 3\sigma/100$

**Lemma A.11.** *For every  $t \in [a + \tau, b - \tau]$ , the distance  $d$  between  $\gamma(t)$  and  $\gamma'$  restricted to  $[\theta(t - \tau), \theta(t + \tau)]$  is bounded above by  $5l' + l + 14\delta$ .*

*Proof.* Let  $t \in [a + \tau, b - \tau]$ . Note that  $2\tau \leq \sigma/4$ . Hence by definition of  $\theta$ ,  $|\theta(t + \tau) - \theta(t - \tau)|$  is at most  $\sigma/3$ . Consequently, the restriction of  $\gamma'$  to the interval  $[\theta(t - \tau), \theta(t + \tau)]$  is a  $(1, l')$ -quasi-geodesic lying in a ball  $B$  of radius  $\sigma$ . In particular, this path is  $(l' + 3\delta)$ -quasi-convex in  $B$ . One can choose  $B$  in such a way that it contains  $\gamma(t - \tau)$ ,  $\gamma(t)$  and  $\gamma(t + \tau)$ . This allow us to use projections on a quasi-convex subset (see Lemma 2.10). We define four points of  $B$  (see Fig. 4).

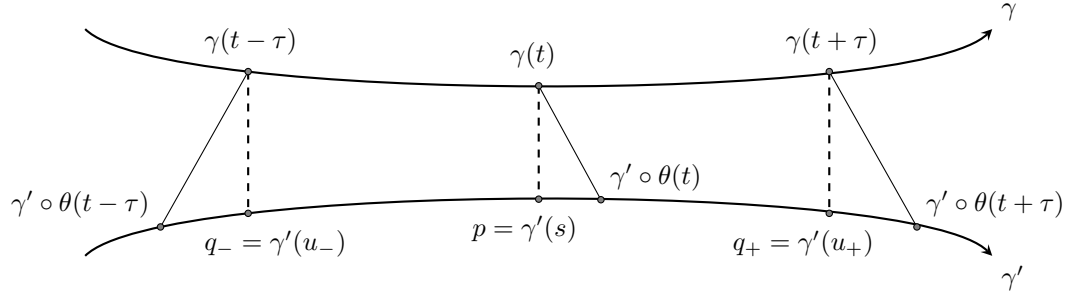


Figure 4: The paths  $\gamma$  and  $\gamma'$

- $p = \gamma'(s)$  is a projection of  $\gamma(t)$  on  $\gamma'([\theta(t - \tau), \theta(t)])$ ,
- $q_- = \gamma'(u_-)$  is a projection of  $\gamma(t - \tau)$  on  $\gamma'([\theta(t - \tau), s])$ ,
- $q_+ = \gamma'(u_+)$  is a projection of  $\gamma(t + \tau)$  on  $\gamma'([s, \theta(t + \tau)])$ .

In particular,

$$d \leq |\gamma(t) - p| < 10^{-2}\sigma \text{ and } |\gamma(t \pm \tau) - q_{\pm}| < 10^{-2}\sigma.$$

**Claim 1.**  $|p - q_-| > \sigma/200$  and  $|p - q_+| > \sigma/200$ . Without loss of generality we can assume that  $s \leq \theta(t)$ . Suppose, contrary to our claim, that  $|p - q_-| \leq \sigma/200$ . Hence the length of  $\gamma'$  restricted to  $[s, u_-]$  is at most  $\sigma/200 + l'$ . However the distances  $|\gamma(t - \tau) - q_-|$  and  $|\gamma(t) - p|$  are bounded above by  $\sigma/100$ . Therefore there exists a path  $\gamma_0$  joining  $\gamma(t - \tau)$  to  $\gamma(t)$  whose length is at most  $\sigma/50 + l'$  and which follows  $\gamma'$  restricted to  $[s, u_-]$ . On the other hand  $|\theta(t - \tau) - u_-|$  and  $|\theta(t) - s|$  are bounded above by  $\sigma/20 + l'$ . It follows from Lemma A.5 that  $\gamma$  restricted to  $[a, t - \tau]$  and  $\gamma$  restricted to  $[t, b]$  respectively follow  $\gamma'$  restricted to  $[a', u_-]$  and  $\gamma'$  restricted to  $[s, b']$ . According to Lemma A.4, concatenating  $\gamma$  restricted to  $[a, t - \tau]$ ,  $\gamma_0$  and  $\gamma$  restricted to  $[t, b]$  gives a path

joining  $x$  to  $y$  which follows  $\gamma'$ . In particular, it belongs to  $\mathcal{C}$ . Nevertheless, its length is bounded above by

$$L\left(\gamma|_{[a, t-\tau]}\right) + L\left(\gamma|_{[t, b]}\right) + \frac{\sigma}{50} + l' = L(\gamma) - \tau + \frac{\sigma}{50} + l' < L(\gamma) - l,$$

which contradicts the minimality of  $\gamma$ . With a similar argument, we prove that

$$|\gamma' \circ \theta(t) - q_+| > \sigma/200 + l'.$$

However  $\gamma'$  restricted to  $[\theta(t-\tau), \theta(t+\tau)]$  is a  $(1, l')$ -quasi-geodesic and  $s \leq \theta(t)$ . It follows that  $|p - q_+| > \sigma/200$ .

**Claim 2.** The length of  $\gamma'$  restricted to  $[u_-, u_+]$  is bounded above by

$$L\left(\gamma|_{[t-\tau, t+\tau]}\right) - |\gamma(t-\tau) - q_-| - |\gamma(t+\tau) - q_+| - 2d + 9l' + 28\delta.$$

The points  $p$  and  $q_-$  are respective projections of  $\gamma(t)$  and  $\gamma(t-\tau)$  on the path  $\gamma'$  restricted to  $[\theta(t-\tau), s]$ . By projection on a quasi-convex, we have

$$|p - q_-| \leq \max \left\{ |\gamma(t-\tau) - \gamma(t)| - |\gamma(t-\tau) - q_-| - |\gamma(t) - p| + 2\varepsilon, \varepsilon \right\}$$

where  $\varepsilon = 2l' + 7\delta$ . According to our previous claim, we necessarily have

$$|\gamma(t-\tau) - \gamma(t)| \geq |\gamma(t-\tau) - q_-| + |q_- - p| + |p - \gamma(t)| - 4l' - 14\delta.$$

In the same way we obtain

$$|\gamma(t+\tau) - \gamma(t)| \geq |\gamma(t+\tau) - q_+| + |q_+ - p| + |p - \gamma(t)| - 4l' - 14\delta.$$

Combining these two inequalities, we see that the length of  $\gamma$  restricted to  $[t-\tau, t+\tau]$  is bounded below by

$$|\gamma(t-\tau) - q_-| + |q_- - q_+| + |q_+ - \gamma(t+\tau)| + 2d - 8l' - 28\delta.$$

However  $\gamma'$  restricted to  $[t-\tau, t+\tau]$  is a  $(1, l')$ -quasi-geodesic. Therefore,

$$|q_- - q_+| \geq L\left(\gamma'|_{[u_-, u_+]}\right) - l'.$$

Combined with the previous inequality we obtain our second claim.

Recall that  $|\gamma(t \pm \tau) - q_{\pm}| < \sigma/100$ . Hence there exists a path  $\gamma_1$  joining  $\gamma(t-\tau)$  to  $\gamma(t+\tau)$  which follows the restriction of  $\gamma'$  to  $[u_-, u_+]$  and whose length is at most

$$|\gamma(t-\tau) - q_-| + L\left(\gamma'|_{[u_-, u_+]}\right) + |q_+ - \gamma(t+\tau)| + l.$$

According to our second claim

$$L(\gamma_1) \leq L\left(\gamma|_{[t-\tau, t+\tau]}\right) - 2d + 9l' + l + 28\delta.$$

On the other hand, by Lemma A.5, the restriction of  $\gamma$  to  $[a, t-\tau]$  (respectively  $[t+\tau, b]$ ) follows the one of  $\gamma'$  to  $[a', u_-]$  (respectively  $[u_+, b']$ ). By concatenating  $\gamma$  restricted to  $[a, t-\tau]$ ,  $\gamma_1$  and  $\gamma$  restricted to  $[t+\tau, b]$  we obtain a path joining  $x$  to  $y$  which follows  $\gamma'$  (see Lemma A.4). In particular, this is an element of  $\mathcal{C}$ . Moreover, its length is bounded above by  $L(\gamma) - 2d + 9l' + l + 28\delta$ . By construction,  $\gamma$  minimizes up to  $l$  the length of the elements of  $\mathcal{C}$ , hence  $d \leq 5l' + l + 14\delta$ .  $\square$

**Lemma A.12.** *The path  $\gamma$   $D$ -follows  $\gamma'$  with*

$$D = \max\{|x - x'|, |y - y'|, 5l' + l + 14\delta\} + 3l + 3l' + 8\delta < \sigma/100.$$

*Proof.* We consider the subdivision  $a = t_0 \leq \dots \leq t_m = b$  of  $[a, b]$  such that for every  $i \in \{0, \dots, m-2\}$ ,  $|t_{i+1} - t_i| = 2\tau$  and  $2\tau \leq |t_m - t_{m-1}| \leq 4\tau$ . We claim that for every  $i \in \{0, \dots, m-1\}$ , the restriction of  $\gamma$  to  $[t_i, t_{i+1}]$  is a  $(1, l)$ -quasi-geodesic. According to Lemma A.11, there exists a subdivision  $a' = s_0 \leq \dots \leq s_m = b'$  of  $[a', b']$  satisfying the following properties.

(i) For every  $i \in \{0, \dots, m\}$ ,

$$|\gamma(t_i) - \gamma'(s_i)| \leq \max\{|x - x'|, |y - y'|, 5l' + l + 14\delta\} < \sigma/200.$$

(ii) For every  $i \in \{0, \dots, m-1\}$ ,

$$|s_{i+1} - s_i| \leq |t_{i+1} - t_i| + \frac{\sigma}{50} + l' \leq \frac{\sigma}{3}.$$

Let  $\eta \in (0, 10^{-10}\sigma)$ . Let  $i \in \{0, \dots, m-1\}$ . There exists a  $(1, \eta)$ -quasi-geodesic  $c_i$  joining  $\gamma(t_i)$  to  $\gamma(t_{i+1})$ . Thus  $c_i$  and  $\gamma'$  restricted to  $[t_i, t_{i+1}]$  are respectively  $(1, \eta)$ - and  $(1, l')$ -quasi-geodesics contained in a ball of radius  $\sigma$ . By Lemma A.6, they follow each other. Consequently, the path  $c$ , obtained by concatenating the  $c_i$ 's, follows  $\gamma'$  (see Lemma A.4). In particular,  $c$  belongs to  $\mathcal{C}$ . Therefore its length is bounded below by  $L(\gamma) - l$ . Since the  $c_i$ 's are  $(1, \eta)$ -quasi-geodesics one has

$$\sum_{i=0}^{m-1} L(\gamma|_{[t_i, t_{i+1}]}) \leq \sum_{i=0}^{m-1} |\gamma(t_{i+1}) - \gamma(t_i)| + m\eta + l.$$

It implies that for every  $i \in \{0, \dots, m-1\}$ , the length of  $\gamma$  restricted to  $[t_i, t_{i+1}]$  is bounded above by  $|\gamma(t_{i+1}) - \gamma(t_i)| + m\eta + l$ . This inequality holds for every  $\eta > 0$ . Hence the restriction of  $\gamma$  to  $[t_i, t_{i+1}]$  is a  $(1, l)$ -quasi-geodesic, which proves our claim.

As we did previously with the  $c_i$ 's, we can prove that for every  $i \in \{0, \dots, m-1\}$ , the paths  $\gamma$  and  $\gamma'$  respectively restricted to  $[t_i, t_{i+1}]$  and  $[s_i, s_{i+1}]$   $D$ -follows each other where

$$D = \max\{|x - x'|, |y - y'|, 5l' + l + 14\delta\} + 3l + 3l' + 8\delta.$$

Hence by Lemma A.4  $\gamma$   $D$ -follows  $\gamma'$ . □

We can now finish the proof by showing that  $\gamma$  is a  $\sigma/3$ -local  $(1, l)$ -quasi-geodesic. With the notations of the previous lemma,  $\gamma$   $D$ -follows  $\gamma'$ . In particular, it provides a map  $\mu : [a, b] \rightarrow [a', b']$  (possibly different from  $\theta$ ) which satisfies the axioms of Definition A.3. Let  $s, t \in [a, b]$ ,  $s \leq t$  such that  $|s - t| \leq \sigma/3$ . Since  $\gamma'$  is a  $\sigma/3$ -local  $(1, l')$ -quasi-geodesic, we have

$$|\mu(s) - \mu(t)| \leq |s - t| + 4D + 2l' \leq \frac{\sigma}{2}.$$

Therefore  $\gamma'$  restricted to  $[\mu(s), \mu(t)]$  is contained in a ball  $B$  of radius  $\sigma$ . By hyperbolicity, this path is a  $(1, l' + 2\delta)$ -quasi-geodesic in  $B$ . Let  $\eta \in (0, 10^{-10}\sigma)$  and  $\gamma_2$  be a  $(1, \eta)$ -quasi-geodesic joining  $\gamma(s)$  to  $\gamma(t)$ . Note that we can choose  $B$  in such a way that it also contains  $\gamma_2$ . Recall that

$|\gamma(s) - \gamma' \circ \mu(s)| < D$ . The same holds for  $t$ . Hence by Lemma A.6,  $\gamma_2$  follows  $\gamma'$  restricted to  $[\mu(s), \mu(t)]$ . Now we consider the concatenation of  $\gamma$  restricted to  $[a, s]$ ,  $\gamma_2$  and  $\gamma$  restricted to  $[t, b]$ , we obtain a path which follows  $\gamma'$  joining  $x$  to  $y$  (see Lemma A.4). Consequently, it belongs to  $\mathcal{C}$  and its length is bounded below by  $L(\gamma) - l$ . However  $\gamma_2$  is a  $(1, \eta)$ -quasi-geodesic, thus

$$L(\gamma) \leq L(\gamma|_{[a, s]}) + |\gamma(s) - \gamma(t)| + L(\gamma|_{[t, b]}) + l + \eta.$$

It implies that the length of  $\gamma$  restricted to  $[s, t]$  is less than  $|\gamma(s) - \gamma(t)| + \eta + l$ . This inequality holds for every  $\eta > 0$  and  $s, t \in [a, b]$  such that  $|s - t| \leq \sigma/3$ . Consequently,  $\gamma$  is a  $\sigma/3$ -locally  $(1, l)$ -quasi-geodesic path.  $\square$

#### A.4 The space of quasi-geodesic paths.

The goal of this section is to prove the stability of locally quasi-geodesic paths in  $X$ . More precisely prove the following result

**Proposition A.13.** *Let  $l \in (0, 10^{-5}\sigma)$ . Let  $x, y$  and  $y'$  be three points of  $X$  such that  $|y - y'| < 10^{-5}\sigma$ . Let  $\gamma$  (respectively  $\gamma'$ ) be a  $\sigma/3$ -locally  $(1, l)$ -quasi-geodesic path joining  $x$  to  $y$  (respectively to  $y'$ ). Then the Hausdorff distance between  $\gamma$  and  $\gamma'$  is less than  $\max\{|y - y'|, 2l + 5\delta\} + 6l + 8\delta$ .*

Let  $l \in (0, 10^{-5}\sigma)$ . We fix a base point  $x_0 \in X$  and denote by  $\Gamma$  the set of  $\sigma/3$ -local  $(1, l)$ -quasi-geodesics starting at  $x_0$ . We endow  $\Gamma$  with the equivalence relation defined previously: two elements  $\gamma$  and  $\gamma'$  of  $\Gamma$  are equivalent if they have the same terminal points and they follow each other. The set of equivalence classes is denoted by  $\tilde{X}$ . Given a path  $\gamma \in \Gamma$  we write  $[\gamma]$  for its equivalence class. By convention  $\tilde{x}_0$  stands for the point of  $\tilde{X}$  represented by the constant path equal to  $x_0$ . The map  $p : \Gamma \rightarrow X$  is defined to send every path to its terminal point. It induces a map from  $\tilde{X}$  to  $X$  that we still denote by  $p$ . We are going to prove that  $p : \tilde{X} \rightarrow X$  is a path-connected cover of  $X$ . To that end we need first to define a topology on  $\tilde{X}$ .

**Topology of  $\tilde{X}$ .** Let  $\tilde{x} = [\gamma]$  be a point of  $\tilde{X}$ . Let  $\varepsilon \in (0, 10^{-5}\sigma)$ . We define  $U_{\tilde{x}, \varepsilon}$  as the set of equivalence classes  $\tilde{x}' = [\gamma'] \in \tilde{X}$  such that  $\gamma'$  follows  $\gamma$  and conversely and  $|p(\tilde{x}) - p(\tilde{x}')| < \varepsilon$ . According to Proposition A.8,  $U_{\tilde{x}, \varepsilon}$  is well-defined and does not depend on the choice of the path  $\gamma$  representing  $\tilde{x}$ .

**Lemma A.14.** *The collection  $\{U_{\tilde{x}, \varepsilon}\}$  where  $\tilde{x} \in \tilde{X}$  and  $\varepsilon \in (0, 10^{-5}\sigma)$  forms a base of open sets of  $\tilde{X}$ .*

*Proof.* First note that for every  $\tilde{x} \in \tilde{X}$  and  $\varepsilon \in (0, 10^{-5}\sigma)$ ,  $\tilde{x}$  belongs to  $U_{\tilde{x}, \varepsilon}$ . Therefore  $\{U_{\tilde{x}, \varepsilon}\}$  covers  $\tilde{X}$ . Let  $\tilde{y} = [\nu]$  be a point of  $U_{\tilde{x}, \varepsilon} \cap U_{\tilde{x}', \varepsilon'}$  where  $\tilde{x} = [\gamma]$ ,  $\tilde{x}' = [\gamma']$  are two points of  $\tilde{X}$  and  $\varepsilon, \varepsilon' \in (0, 10^{-5}\sigma)$ . By definition there exists  $\eta \in (0, 10^{-5}\sigma)$  such that  $|p(\tilde{y}) - p(\tilde{x})| + \eta < \varepsilon$  and  $|p(\tilde{y}) - p(\tilde{x}')| + \eta < \varepsilon'$ . We are going to prove that  $U_{\tilde{y}, \eta}$  is contained in  $U_{\tilde{x}, \varepsilon} \cap U_{\tilde{x}', \varepsilon'}$ . Let  $\tilde{z} = [c]$  be a point of  $U_{\tilde{y}, \eta}$ . By definition  $c$  follows  $\nu$  and conversely. Moreover,  $|p(\tilde{z}) - p(\tilde{y})| < \eta$ . However  $\tilde{y}$  lies in  $U_{\tilde{x}, \varepsilon}$ . Therefore  $\nu$  follows  $\gamma$  and conversely. Moreover,  $|p(\tilde{y}) - p(\tilde{x})| < \varepsilon$ . By Proposition A.8,  $c$  follows  $\gamma$  and conversely. Furthermore

$$|p(\tilde{z}) - p(\tilde{x})| \leq |p(\tilde{z}) - p(\tilde{y})| + |p(\tilde{y}) - p(\tilde{x})| < |p(\tilde{y}) - p(\tilde{x})| + \eta < \varepsilon.$$

Consequently,  $\tilde{z}$  belongs to  $U_{\tilde{x}, \varepsilon}$ . We prove in the same way that  $\tilde{z} \in U_{\tilde{x}', \varepsilon'}$ , which establishes our claim.  $\square$

**Lemma A.15.** *The space  $\tilde{X}$  is path-connected.*

*Proof.* Let  $\tilde{x}$  be a point of  $\tilde{X}$ . We choose  $\gamma : [a, b] \rightarrow X$  an element of  $\Gamma$  representing  $\tilde{x}$ . We define a path of  $X$  joining  $\tilde{x}_0$  to  $\tilde{x}$  as follows.

$$\begin{aligned} F : [a, b] &\rightarrow \tilde{X}, \\ t &\rightarrow [\gamma|_{[a, t]}]. \end{aligned}$$

We claim that  $F$  is continuous. Let  $t \in [a, b]$  and  $\varepsilon \in (0, 10^{-5}\sigma)$ . Let  $s \in [a, b]$  such that  $|s - t| < \varepsilon$ . Since  $\gamma$  is parametrized by arclength  $|\gamma(s) - \gamma(t)| < \varepsilon$ . By Lemmas A.4 and A.5 the paths  $\gamma$  restricted to  $[a, s]$  and  $[a, t]$  follow each other. Moreover,  $|p \circ F(s) - p \circ F(t)| = |\gamma(s) - \gamma(t)| < \varepsilon$ . Hence  $F(s)$  belongs to  $U_{F(t), \varepsilon}$  which proves that  $F$  is continuous. Every point of  $\tilde{X}$  can be connected to  $\tilde{x}_0$  by a continuous path, thus  $\tilde{X}$  is path-connected.  $\square$

**Lemma A.16.** *The map  $p$  is continuous.*

*Proof.* Let  $x \in X$  and  $\varepsilon \in (0, 10^{-5}\sigma)$ . By construction  $p^{-1}(B(x, \varepsilon))$  is exactly the union of all  $U_{\tilde{x}, \varepsilon}$  where  $\tilde{x} \in p^{-1}(x)$ . Consequently,  $p$  is continuous.  $\square$

**Lemma A.17.** *Let  $\tilde{x} \in \tilde{X}$  and  $\varepsilon \in (0, 10^{-5}\sigma)$ . Then  $p$  induces a homeomorphism from  $U_{\tilde{x}, \varepsilon}$  onto  $B(p(\tilde{x}), \varepsilon)$ .*

*Proof.* By construction, the image by  $p$  of  $U_{\tilde{x}, \varepsilon}$  is contained in  $B(p(\tilde{x}), \varepsilon)$ . We first prove the converse inclusion. To that end we denote by  $\gamma$  a path of  $\Gamma$  representing  $\tilde{x}$ . Let  $y \in B(p(\tilde{x}), \varepsilon)$ . The path  $\gamma$  is a  $\sigma/3$ -locally  $(1, l)$ -quasi-geodesic. Moreover,  $|p(\tilde{x}) - y| < \varepsilon$ . By Proposition A.10 there exists a  $\sigma/3$ -locally  $(1, l)$ -quasi-geodesic  $\nu$  joining  $x_0$  to  $y$  which follows  $\gamma$ . Applying Proposition A.7,  $\gamma$  also follows  $\nu$ . Therefore  $\nu$  defines a point  $\tilde{y} \in U_{\tilde{x}, \varepsilon}$  whose image by  $p$  is  $y$ . Consequently,  $p$  maps  $U_{\tilde{x}, \varepsilon}$  onto  $B(p(\tilde{x}), \varepsilon)$ .

Let  $\tilde{y} = [\nu]$  and  $\tilde{y}' = [\nu']$  be two points  $U_{\tilde{x}, \varepsilon}$  such that  $p(\tilde{y}) = p(\tilde{y}')$ . By construction  $\nu$  (respectively  $\nu'$ ) follows  $\gamma$  and conversely. Moreover,  $|p(\tilde{y}) - p(\tilde{x})| < \varepsilon$  and  $|p(\tilde{y}') - p(\tilde{x})| < \varepsilon$ . By Proposition A.8  $\nu$  and  $\nu'$  follow each other. By assumption they also have the same extremities. It follows that  $\nu$  and  $\nu'$  are equivalent i.e.,  $\tilde{y} = \tilde{y}'$ . Thus the restriction of  $p$  to  $U_{\tilde{x}, \varepsilon}$  is one-to-one. Consequently,  $p$  induces a bijection from  $U_{\tilde{x}, \varepsilon}$  onto  $B(x, \varepsilon)$ . To complete the proof we have to show that this bijection is a homeomorphism. Since  $p$  is continuous, we only need to show that it is open. To that end it is sufficient to observe that for every  $U_{\tilde{y}, \eta}$  contained in  $U_{\tilde{x}, \varepsilon}$ ,  $p(U_{\tilde{y}, \eta})$  is an open set which is a consequence of the first part of the proof.  $\square$

**Proposition A.18.** *The map  $p : \tilde{X} \rightarrow X$  is a covering map.*

*Proof.* Every point of  $x$  of  $X$  can be joined to  $x_0$  by a  $(1, l)$ -quasi-geodesic. Therefore  $p$  is onto. We already proved that  $p$  is a continuous map which induces for every  $\tilde{x} \in \tilde{X}$  and every  $\varepsilon \in (0, 10^{-5}\sigma)$  a homeomorphism from  $U_{\tilde{x}, \varepsilon}$  onto  $B(p(\tilde{x}), \varepsilon)$  (Lemma A.16 and Lemma A.17). Therefore it is sufficient to prove that for all  $\tilde{x}, \tilde{x}' \in \tilde{X}$  such that  $p(\tilde{x}) = p(\tilde{x}')$  for every  $\varepsilon \in (0, 10^{-5}\sigma)$ , if  $U_{\tilde{x}, \varepsilon} \cap U_{\tilde{x}', \varepsilon}$  is non-empty then  $\tilde{x} = \tilde{x}'$ . We denote by  $\gamma$  and  $\gamma'$  paths of  $\Gamma$  respectively representing  $\tilde{x}$  and  $\tilde{x}'$ . Assume that there exists  $\tilde{y} = [\nu]$  a point in  $U_{\tilde{x}, \varepsilon} \cap U_{\tilde{x}', \varepsilon}$ . By definition  $\nu$  follows  $\gamma$  (respectively  $\gamma'$ ) and conversely. Moreover,  $|p(\tilde{y}) - p(\tilde{x})| < \varepsilon$  and  $|p(\tilde{y}) - p(\tilde{x}')| < \varepsilon$ . According to Proposition A.8  $\gamma$  and  $\gamma'$  follow each other. In addition they have the same extremities. Consequently, they are equivalent, hence  $\tilde{x} = \tilde{x}'$ .  $\square$



*Proof of Proposition A.13.* Being a covering map,  $p$  induces a one-to-one homomorphism  $p_* : \pi_1(\tilde{X}) \rightarrow \pi_1(X)$ . We claim that this map is actually onto. By assumption  $X$  is  $10^{-5}\sigma$ -simply connected. Hence its fundamental group is normally generated by the set of homotopy classes of loops of diameter less than  $10^{-5}\sigma$ . Let  $c$  be such a loop. An element  $g \in \pi_1(X)$  of the conjugacy class represented by  $c$  is characterized by a path  $c_0$  joining  $x_0$  to a point of  $c$ . Since  $p$  is a covering map,  $c_0$  can be lifted as a path  $\tilde{c}_0$  starting at  $\tilde{x}_0$ . We denote by  $\tilde{x}$  the terminal point of  $\tilde{c}_0$ . Hence  $p(\tilde{x}) = x$ . However  $p$  induces an homeomorphism from  $U_{\tilde{x}, 10^{-5}\sigma}$  onto  $B(x, 10^{-5}\sigma)$  (Lemma A.17). Since  $c$  lies in  $B(x, 10^{-5}\sigma)$ , it can be lifted as a loop  $\tilde{c}$  in  $\tilde{X}$ . The paths  $\tilde{c}_0$  and  $\tilde{c}$  defines an element  $\tilde{g} \in \pi_1(\tilde{X})$  which is sent on  $g$  by  $p_*$ . Hence  $p_*$  is onto and therefore an isomorphism. It follows that  $p$  is a bijection. Let  $\gamma$  and  $\gamma'$  be two  $\sigma/3$ -locally  $(1, l)$ -quasi-geodesic starting at  $x_0$ . We assume that their respective endpoints  $y$  and  $y'$  satisfies  $|y - y'| < 10^{-5}\sigma$ . Hence  $\tilde{y} = [\gamma]$  and  $\tilde{y}' = [\gamma']$  are two points of  $\tilde{X}$  such that  $|p(\tilde{y}) - p(\tilde{y}')| < 10^{-5}\sigma$ . Since  $p$  is a bijection, Lemma A.17 implies that  $\tilde{y}'$  belongs to  $U_{\tilde{y}, 10^{-5}\sigma}$ . In particular,  $\gamma$  and  $\gamma'$  follow each other. However by Lemma A.7 they actually  $D$ -follow each other where  $D = \max\{|y - y'|, 2l + 5\delta\} + 6l + 8\delta$ . Therefore the Hausdorff distance between them is less than  $D$ . Note that no constant in the proof depends on the choice of the base point  $x_0$ . Therefore the results holds for any two  $\sigma/3$ -locally  $(1, l)$ -quasi-geodesics with the same initial point.  $\square$

## A.5 Global hyperbolicity of $X$

**Proposition A.19.** *Let  $l \in (0, 10^{-10}\sigma)$ . Let  $x, y$  and  $y'$  be three points of  $X$ . Let  $\gamma$  (respectively  $\gamma', \nu$ ) be a  $(1, l)$ -quasi-geodesic joining  $x$  to  $y$  (respectively  $x$  to  $y'$ ,  $y$  to  $y'$ ). Then  $\nu$  is contained in the  $\Delta$ -neighborhood of  $\gamma \cup \gamma'$  where  $\Delta = 50l + 96\delta$ .*

*Proof.* We denote by  $\gamma : [a, b] \rightarrow X$  and  $\gamma' : [a', b'] \rightarrow X$  parametrization by arclength of  $\gamma$  and  $\gamma'$ . We define  $s$  to be the largest number in  $[a, b]$  such that  $d(\gamma(s), \gamma') \leq 4l + 5\delta$ . Assume first that  $s = b$ . Therefore  $y = \gamma(b)$  is  $(4l + 5\delta)$ -close to  $\gamma'$ . Applying Proposition A.13,  $\nu$  lies in the  $(10l + 13\delta)$ -neighborhood of  $\gamma'$ .

Assume now that  $s < b$ . It follows that  $d(\gamma(s), \gamma') = 4l + 5\delta$ . We write  $p = \gamma(s)$ . Let  $p' = \gamma'(s')$  be a projection of  $p$  on  $\gamma'$ . We put  $t = \min\{b, s + \sigma/3\}$ ,  $t' = \min\{b', s' + \sigma/3\}$ ,  $q = \gamma(t)$  and  $q' = \gamma'(t')$ . See Figure 5. The restriction of  $\gamma'$  to  $[s', t']$  and the point  $p$  are simultaneously contained in a ball  $B$  of radius  $\sigma$ . Hence by Proposition 2.4  $\langle p, q' \rangle_{p'} \leq l + 2\delta$ . On the other hand, by construction of  $s$ , for every  $u \in [s, b]$ ,

$$|p - p'| = d(p, \gamma') \leq d(\gamma(u), \gamma') \leq |\gamma(u) - p'|.$$

Hence  $p$  is a projection of  $p'$  on  $\gamma$ . Reasoning as previously, we get  $\langle p', q \rangle_p \leq l + 2\delta$ . The four points  $p, q, p'$  and  $q'$  are contained in a ball of radius  $\sigma$ . Hence, by hyperbolicity,

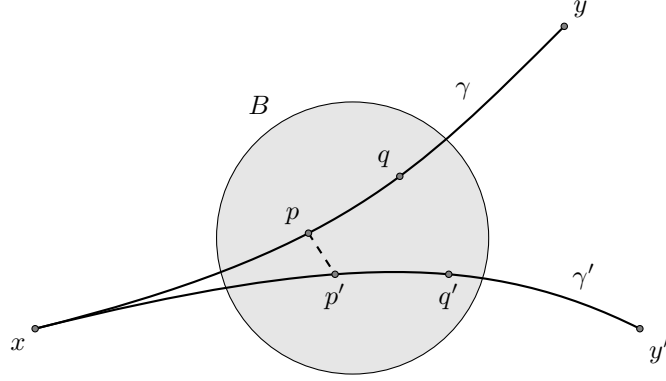
$$|q - p'| + |q' - p| \leq \max\{|p - p'| + |q - q'|, |p - q| + |p' - q'|\} + 2\delta.$$

Combined with the upper bound on the Gromov products, it leads to

$$|p - q| + |p' - q'| + 2|p - p'| \leq \max\{|p - p'| + |q - q'|, |p - q| + |p' - q'|\} + 4l + 10\delta.$$

Recall that  $|p - p'| = 4l + 5\delta$ . Consequently, the maximum cannot be achieved by the second term. Hence

$$|q - q'| \geq |q - p| + |p - p'| + |p' - q'| - 4l - 10\delta. \quad (22)$$

Figure 5: Thin “geodesic” triangle in  $X$ 

Let  $\eta \in (0, 10^{-10}\sigma)$ . We denote by  $\nu_0$  be a  $(1, \eta)$ -quasi-geodesic joining  $p$  to  $p'$ . We now define a path  $\nu'$  by concatenating  $\gamma$  restricted to  $[s, b]$  (with the reverse parametrization),  $\nu_0$  and  $\gamma'$  restricted to  $[s', b']$ . Using (22), the length of the portion of  $\nu'$  between  $q$  and  $q'$  is bounded above by

$$|q - p| + |p - p'| + |p' - q'| + 2l + \eta \leq |q - q'| + 6l + \eta + 10\delta.$$

Therefore it is a  $(1, l')$ -quasi-geodesic with  $l' = 6l + \eta + 10\delta$ . On the other hand the portion of  $\nu$  between  $p$  and  $y$  (respectively  $p'$  and  $y'$ ) is a  $\sigma/3$ -local  $(1, l)$ -quasi-geodesic. Note that if  $t < b$  then  $|s - t| = \sigma/3$ . The same holds for  $t'$ . Hence  $\nu'$  is a  $\sigma/3$ -local  $(1, l')$ -quasi-geodesic joining  $y$  and  $y'$ . So is also  $\nu$ . Hence by Proposition A.13, the Hausdorff distance between  $\nu$  and  $\nu'$  is less than  $48l + 8\eta + 93\delta$ . However, by construction  $\nu'$  is contained in the  $(2l + 3\delta + \eta)$ -neighborhood of  $\gamma \cup \gamma'$ . Hence  $\nu$  lies in the  $(50l + 9\eta + 96\delta)$ -neighborhood of  $\gamma \cup \gamma'$ . This estimate holds for every  $\eta > 0$  which leads to the result.  $\square$

**Corollary A.20.** *Let  $x, y$  and  $y'$  be three points of  $X$ . Let  $l \in (0, 10^{-10}\sigma)$ . Let  $\nu$  be a  $(1, l)$ -quasi-geodesic joining  $y$  to  $y'$ . Then*

$$\langle y, y' \rangle_x - \frac{1}{2}l \leq d(x, \nu) \leq \langle y, y' \rangle_x + 101l + 192\delta.$$

*Proof.* Let  $\nu : [a, b] \rightarrow X$  be a parametrization by arclength of  $\nu$ . For every  $t \in [a, b]$ , the triangle inequality gives

$$\langle y, y' \rangle_x \leq |x - \nu(t)| + \langle y, y' \rangle_{\nu(t)} \leq |x - \nu(t)| + \frac{1}{2}l.$$

Hence  $\langle y, y' \rangle_x \leq d(x, \nu) + l/2$ , which gives the first inequality. Let  $\gamma$  and  $\gamma'$  be  $(1, l)$ -quasi-geodesics respectively joining  $x$  to  $y$  and  $x$  to  $y'$ . By Proposition A.19,  $\nu$  lies in the  $(50l + 96\delta)$ -neighborhood of  $\gamma \cup \gamma'$ . Therefore there exists  $t \in [a, b]$  such that  $d(\nu(t), \gamma) \leq 50l + 96\delta$  and  $d(\nu(t), \gamma') \leq 50l + 96\delta$ . We denote by  $u$  and  $u'$  respective projections of  $p$  on  $\gamma$  and  $\gamma'$ . The triangle inequality leads to

$$\langle y, y' \rangle_x \geq |\nu(t) - x| - |\nu(t) - u| - |\nu(t) - u'| - \langle x, y \rangle_u - \langle x, y' \rangle_{u'}.$$

Thus  $\langle y, y' \rangle_x \geq d(x, \nu) - 101l - 192\delta$ .  $\square$

**Corollary A.21.** *The space  $X$  is  $300\delta$ -hyperbolic.*

*Proof.* Let  $x, y, y'$  and  $t$  be four points of  $X$ . Let  $l \in (0, 10^{-10}\sigma)$ . We denote by  $\gamma$  (respectively  $\gamma', \nu$ ) a  $(1, l)$ -quasi-geodesic joining  $x$  to  $y$  (respectively  $x$  to  $y'$ ,  $y$  to  $y'$ ). Let  $p$  be a projection of  $t$  on  $\nu$ . By Proposition A.19,  $p$  lies in the  $(50l + 96\delta)$ -neighborhood of  $\gamma \cup \gamma'$ . Thus the triangle inequality gives

$$d(t, \nu) = |t - p| \geq \min \{d(t, \gamma), d(t, \gamma')\} - 50l - 96\delta.$$

Combined with Corollary A.20 it yields

$$\langle y, y' \rangle_t \geq \min \{ \langle y, x \rangle_t, \langle y', x \rangle_t \} - 152l - 288\delta.$$

This inequality holds for every  $l > 0$ , thus  $\langle y, y' \rangle_t \geq \min \{ \langle y, x \rangle_t, \langle y', x \rangle_t \} - 288\delta$ . Hence  $X$  is  $288\delta$ -hyperbolic.  $\square$

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